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Example: Let's consider our experiment of flipping three coins. Assuming these coins are fair, we have 8 equally-likely outcomes with probabilities $\frac{1}{8}$.

Thus, for E_1 the event that exactly two heads come up, i.e.

$$E_1 = \{HHT, HTH, THH\},$$

we have that

$$p(E_1) = p(HHT) + p(HTH) + p(THH)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

Similarly, for E_2 the event that tails comes up first on the second flip, we have

$$E_2 = \{HTH, HTT\},$$

whereby $p(E_2) = p(HTH) + p(HTT)$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Similarly, as $E_1 \cap E_2 = \{HTH\}$ and $E_1 \cup E_2 = \{HHT, HTH, THH, HTT\}$,

we have that $p(E_1 \cap E_2) = \frac{1}{8}$

and $p(E_1 \cup E_2) = \frac{4}{8} = \frac{1}{2}.$

Note: For E any one of the four events above,

we have $p(E) = \dfrac{|E|}{|\Omega|}$.

This is always the case when Ω (our sample space of all possible outcomes) is made up of equally-likely outcomes.

<u>Why?</u> If $p(\omega_i) = p$ for all ω_i,

we have
$$1 = p(\Omega) = p(\omega_1) + p(\omega_2) + \cdots + p(\omega_n)$$
$$= \underbrace{p + p + \cdots + p}_{|\Omega| \text{ times}},$$

whereby $p = \dfrac{1}{|\Omega|}$ and thus

$$p(E) = \underbrace{p + p + \cdots + p}_{|E| \text{ times}} = |E| \cdot p,$$

whereby $p(E) = \dfrac{|E|}{|\Omega|}$.

Thus, determining $p(E)$ is just a counting problem in the case where all outcomes are equally likely.

- For an example where outcomes are not equally likely, consider the experiment of rolling and

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summing two (fair) dice.

As the dice are fair, we have equally-likely dice-roll outcomes.

There are 36 such outcomes by the Product Principle:

They can be represented using ordered pairs:

$$(1,1), (1,2), \ldots, (1,6)$$
$$(2,1), (2,2), \ldots, (2,6)$$
$$\vdots$$
$$(6,1), (6,2), \ldots, (6,6).$$

Thus the probability of one of these outcomes is $1/36$.

Now, our experiment consists of rolling & summing the dice.

Here $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Here, the outcome 2 has probability $1/36$ as it corresponds to the dice-roll outcome $(1,1)$, whereas the outcome 3 has probability $2/36 = 1/18$ as it corresponds to the dice-roll outcomes $(1,2)$ and $(2,1)$.

See Tutorial Sheet 4 for more details.

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Given that the probability of an event is the sum of the probabilities of the outcomes that make up that event,

i.e. $p(E) = p(\omega_1) + \cdots + p(\omega_k)$

for $E = \{\omega_1, \cdots, \omega_k\}$,

we can use our knowledge of sets to establish the following:

<u>Addition Rule</u>: For A and B events, we have that

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

In particular, if A and B are <u>mutually exclusive</u> (i.e. $A \cap B = \phi$),

then $p(A \cup B) = p(A) + p(B)$.

<u>Complement Rule</u>: For A an event, we have that $p(\bar{A}) = 1 - p(A)$.

This follows from our Addition Rule, as $\Omega = A \cup \bar{A}$ with $A \cap \bar{A} = \phi$

whereby $1 = p(\Omega) = p(A) + p(\bar{A})$.

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The Complement Rule is useful for calculating $p(A)$ when $\bar{A}$ is a simpler event than A.

Eg. Suppose we flip 3 fair coins & A is the event that at least one head comes up.

Here $\bar{A} = \{TTT\}$, whereby

$$p(\bar{A}) = \tfrac{1}{8} \text{ \& thus}$$

$$p(A) = 1 - p(\bar{A}) = 1 - \tfrac{1}{8} = \tfrac{7}{8}.$$

Eg. Suppose we roll & sum two fair dice. Let A be the event that the sum is at least 4.

Thus $p(A) = p(4) + p(5) + p(6) + \dots + p(12)$

However, $\bar{A} = \{2, 3\}$ and

$$p(\bar{A}) = p(2) + p(3) = \tfrac{1}{36} + \tfrac{2}{36} = \tfrac{1}{12}.$$

Thus $p(A) = 1 - p(\bar{A}) = 1 - \tfrac{1}{12} = \tfrac{11}{12}.$

Conditional Probability

Here, we seek to measure the likelihood of an event A occuring given the knowledge that an event B occurs.

For example, selecting a card at random from a standard deck of 52 cards, our event A might be that the card is a heart, while B might be that the card is red.

We know $p(A) = \frac{13}{52} = \frac{1}{4}$ as we have four equally-likely suits.

Given the knowledge that B happens, the probability of A occurring becomes $\frac{1}{2}$, as there are two equally-likely red suits.

We write $p(A \mid B) = \frac{1}{2}$ to denote this.

Here, $p(B \mid A) = 1$

as A implies B in this case.

Defn: For B an event with $p(B) > 0$, we define the conditional probability of A given B to be

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

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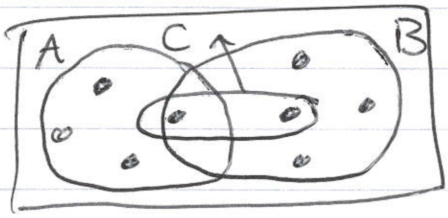
This definition makes sense, as B plays the role of our sample space in this scenario: all possible outcomes must be in B, given that B occurs.

Thus, $A \cap B$ represents the possible outcomes in A.

Suppose we have the following sample space of equally-likely outcomes:



What is $p(A)$? $p(A) = 4/8 = 1/2$

$p(C) = 2/8 = 1/4$

$p(C \mid B) = 2/5$, etc.