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<u>Recall</u>: We define the conditional probability of an event A <u>given</u> that an B occurs as

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

(Here $p(B) > 0$)

<u>Example</u>: Three fair coins are flipped, giving rise to 8 equally-likely outcomes.

Let **B** be that more heads than tails land & **A** be that three heads land.

So $\Omega = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}$,

$A = \{HHH\}$ & $B = \{HHH, HHT, HTH, THH\}$

Now, $p(A) = \underline{\underline{\frac{1}{8}}}$

whereas we have

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} = \frac{1/8}{4/8}$$

$$= \underline{\underline{\frac{1}{4}}}$$

$\left(\text{Recall}: \frac{a/b}{c/d} = \frac{ad}{bc}\right)$

Here, $\bar{A}$ is the event that the result is not three heads.

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i.e. $\bar{A} = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Now $$p(\bar{A} \mid B) = \frac{p(\bar{A} \cap B)}{p(B)} = \frac{3/8}{4/8}$$
$$= 3/4.$$

Note: $p(\bar{A} \mid B) = 1 - p(A \mid B)$ here.

Indeed, more generally, as our conditional probability is a probability measure in its own right, it satisfies the Addition and Complement Rules.

Note: If $p(B_i) > 0$, then by definition

$$p(A \mid B_i) = \frac{p(A \cap B_i)}{p(B_i)},$$

whereby $p(A \cap B_i) = p(A \mid B_i)\, p(B_i)$.

Now suppose $B_1, \ldots, B_n$ is a partition of Ω where $B_1, \ldots, B_n$ are of positive probability

(i.e. $\Omega = B_1 \cup \ldots \cup B_n$ where $B_i \cap B_j = \phi$ for $i \neq j$ and $p(B_i) > 0$ for $i = 1, \ldots, n$)

Then $p(A) = p(A \cap B_1) + p(A \cap B_2) + \cdots + p(A \cap B_n)$ by our Addition Rule, whereby we obtain the <u>Law of Total Probability</u>, that

$$\boxed{p(A) = p(A|B_1)p(B_1) + p(A|B_2)p(B_2) + \cdots + p(A|B_n)p(B_n)}$$

This Law gives us a systematic way of breaking up problems into manageable pieces.

Question 1 An insurance company classifies $1/4$ of its customers as risky (R) and $3/4$ as normal (N). Normal customers have an accident within a year with probability $1/10$, whereas risky ones are twice as likely. What's the probability that a <u>randomly-selected</u> customer has an accident within a year?

<u>Sol$^{\underline{n}}$</u>: Let A be the event that an accident occurs to the customer within a year. Thus $p(A|N) = 1/10$ and $p(A|R) = 2/10$. As R and N partition the customers, we have that

$$p(A) = p(A|N)p(N) + p(A|R)p(R)$$
$$= \left(\tfrac{1}{10}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{2}{10}\right)\left(\tfrac{1}{4}\right)$$
$$= \frac{5}{40} = \frac{1}{8}.$$

Question 2: Suppose 1% of a pop$^{\underline{n}}$ is infected with a certain disease. A medical test correctly diagnoses 98% of infected people as positive and incorrectly diagnoses 5% of those not infected as positive. What's the prob. that a randomly-selected person tests positive?

Sol$^{\underline{n}}$: Let I be the event that the person is infected, whereby I and $\overline{I}$ partition the sample space of infection outcomes. Let $+$ denote the event that the person tests positive.

Then
$$p(+) = p(+\cap I) + p(+\cap \overline{I})$$
$$= p(+|I)p(I) + p(+|\overline{I})p(\overline{I})$$
$$= (.98)(.01) + (.05)(.99)$$
$$= .0593.$$

While it's useful to know the power of a diagnostic test as in the above example, i.e. that $p(+|I) = 0.98$, it is clearly desirable to also know $p(I \mid +)$, the prob. that a person who test's positive is infected. Bayes' Theorem allows us to evaluate this.

How?

We know that, for A & B such that $p(A) > 0$ and $p(B) > 0$, we have

$$p(A \cap B) = p(A \mid B)\, p(B)$$

and $p(B \cap A) = p(B \mid A)\, p(A)$.

As $A \cap B = B \cap A$, we have that

$$p(A \mid B)\, p(B) = p(B \mid A)\, p(A),$$

whereby

$$\boxed{p(B \mid A) = \frac{p(A \mid B)\, p(B)}{p(A)}}$$

Bayes' Theorem

As before, if $B_1, \ldots, B_n$ is a partition of Ω with $p(B_i) > 0$ for $i = 1, \ldots, n$, we may use the Law of Total Probability to rewrite Bayes' Theorem as

$$p(B_i \mid A) = \frac{p(A \mid B_i)\, p(B_i)}{p(A \mid B_1)\, p(B_1) + \cdots + p(A \mid B_n)\, p(B_n)}$$

(since $p(A) = p(A \mid B_1)\, p(B_1) + \cdots + p(A \mid B_n)\, p(B_n)$)

- Let's revisit our insurance example: We know that $p(A|R) = 2/10$. What is $p(R|A)$?

We've seen that

$p(A) = \frac{1}{8}$ by the Law of Total Prob.

Thus

$$p(R|A) = \frac{p(A|R) \cdot p(R)}{p(A)}$$

$$= \frac{\left(\frac{2}{10}\right)\left(\frac{1}{4}\right)}{1/8}$$

$$= \frac{2/40}{1/8} = \frac{16}{40} = 2/5.$$

- Revisiting the medical test example, we know that $p(+|I) = .98$. What is $p(I|+)$?

We've seen that $p(+) = .0593$

So

$$p(I|+) = \frac{p(+|I)\,p(I)}{p(+)}$$

$$= \frac{(.98)(.01)}{.0593} = 0.165.$$

We have the following important definition with respect to conditional probability :

Def^n: Two events A and B are <u>independent</u> if $\boxed{p(A\cap B) = p(A).p(B)}$

If A and B are independent, then

$$p(A|B) = \frac{p(A\cap B)}{p(B)} \overset{\downarrow}{=} \frac{p(A)\,p(B)}{p(B)} = p(A)$$

Ex. 1: Selecting a card at random from a standard deck, let A be the event that the card is an ace & B the event that the card is a spade.

Then $p(A) = \frac{4}{52} = \frac{1}{13}$, $p(B) = \frac{1}{4}$, and $p(A\cap B) = \frac{1}{52}$.

As $p(A\cap B) = p(A).p(B)$, it follows that A and B are independent.

Ex. 2: Two dice are rolled. Let A be that the first dice shows 6 and B that the sum of the 2 dice is ≥ 10.

Then $p(A) = \frac{1}{6}$, $p(B) = p(10) + p(11) + p(12)$

$$\left(\underset{\text{Sum is} \geq 10}{B} = \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}\right)$$

So $p(B) = 6/36$

Now $A \cap B = \{(6,4),(6,5),(6,6)\}$

$\Rightarrow \quad p(A \cap B) = 3/36 = 1/12$

Here $p(A \cap B) = 3/36 \neq \underbrace{p(A)}_{1/6} . \underbrace{p(B)}_{6/36}$

- Let's consider a simple experiment, called a "Bernoulli experiment", that has two outcomes, "Success" S and "Failure" F, with probability of Success equal to p.

→ Suppose we run n identical and independent Bernoulli experiments, with a fixed probability of success p in each experiment.
Eg. n coin flips with $p = 1/2$ for fair coins.

Let X count the number of successes in our n Bernoulli experiments.

X is called a <u>binomial</u> random variable, and is denoted by

$$X \sim Bin(n, p).$$

<u>Eg.</u> Consider such an experiment with $n=4$ Bernoulli trials.

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We have $16 = 2^4$ possible outcomes:

	Outcome	Probability	X
	SSSS	p^4	4
	SSSF	$p^3(1-p)$	3
	SSFS	$p^3(1-p)$	3
	SFSS	$p^3(1-p)$	3
	FSSS	$p^3(1-p)$	3
	SSFF	$p^2(1-p)^2$	2
$\binom{4}{2}=6$	⋮		
	FSSF	$p^2(1-p)^2$	2
	SFFF	$p(1-p)^3$	1
$\binom{4}{1}=4$	⋮		
	FFFS	$p(1-p)^3$	1
	FFFF	$(1-p)^4$	0

In general, the Binomial distribution assigns the probability $p(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ to each of the outcomes [redacted] $X=0, X=1, \ldots, X=n$