① MS115

- We can use truth tables to prove De Morgan's Laws

(i) $\text{not}(P \wedge Q) \equiv \text{not } P \vee \text{not } Q$

$\wedge$ → AND, $\vee$ → OR

(ii) $\text{not}(P \vee Q) \equiv \text{not } P \wedge \text{not } Q$

$\vee$ → OR, $\wedge$ → AND

Let's prove (i):

P	Q	not P	not Q	$P \wedge Q$	$\text{not}(P \wedge Q)$	$\text{not } P \vee \text{not } Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Hence $\text{not}(P \wedge Q) \equiv \text{not } P \vee \text{not } Q$

$\equiv$ ← logically equivalent to

(ii) as exercise.

- Per the tutorial sheet (1), a proposition is logically true if it is true in all cases

eg. the proposition $P \vee \text{not } P$ is logically true:

(2)

P	not P	$P \vee$ not P
T	F	T
F	T	T

Recall: The conditional operator $P \Rightarrow Q$ states that if P is true then Q is also true.

It has truth table

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

For 2 given propositions P and Q, we will want to prove that $P \Rightarrow Q$ is logically true.

Why? Then we know that if P is true, then Q is true.

How do we show $P \Rightarrow Q$ is logically true for 2 given propositions P & Q?

Strategy: show the case where P is true & Q is false cannot occur.

This gives us 3 methods of argument:

Direct argument: Assume P is true, show that Q is true.

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- Contrapositive argument:

Assume that Q is false, show that P is false

(Recall: the contrapositive of $P \Rightarrow Q$ is the logically equivalent statement that not $Q \Rightarrow$ not P

- Proof by contradiction:

Assume P is true and Q is false and derive a contradiction.

Examples: An integer is a whole number that is zero, positive or negative $(0, \pm 1, \pm 2, \ldots)$

An integer x is even if there is an integer k such that
$$x = 2k.$$
An integer y is odd if
$$y = 2n + 1 \text{ for some integer } n.$$

Direct argument to show that
x and y odd integers $\Rightarrow xy$ is odd

- Let $x = 2n+1$ and $y = 2m+1$ for some integers n and m

Then
$$\begin{aligned} xy &= (2n+1)(2m+1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1 \end{aligned}$$

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- Contrapositive argument:
Let's show that x^2 odd $\Rightarrow$ x odd
by showing x even $\Rightarrow$ x^2 even

$\rightarrow$ Let $x = 2k$ for some integer
Then $x^2 = (2k)(2k) = 2(2k^2)$

- Proof by contradiction

$\rightarrow$ read a proof of the fact that x is not a fraction if $x^2 = 2$.

or

$\rightarrow$ read a proof that there are infinitely many prime numbers.

A statement that contains variables can either be true or false depending on the value of the variables.

Natural application: while ($i < 10$)

These statements are called predicates.

When we fix the value of the variable we have a proposition that is true or false.

eg. $P(n)$: n is an integer greater than 3
$P(-1)$ is false, $P(3)$ is false,
$P(n)$ is true for all $n > 3$

eg. $P(n)$: there exists an integer n such that $n^2 = 4$

$P(-2)$ true, $P(2)$ true, $P(n)$ false otherwise

eg. $P(n)$: $n^2 \geq 0$

→ Does this hold for all integers n?

Yes. How do we prove this?

More generally, let $P(n)$ be some predicate that is defined for all positive integers n.

How do we show $P(n)$ is true in all cases?

- <u>Induction</u>

Let $P(n)$ be a predicate that is defined for all $n \geq 1$.

Suppose that (1) $P(1)$ is true.

and (2) for all $n \geq 1$,

$$P(n) \Rightarrow P(n+1) \text{ is true.}$$

Then $P(n)$ is true for all $n \geq 1$.

(6)

Eg. Let's show that

$P(n): 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

is true for all $n \geq 1$.

Induction : (1) Let's show $P(1)$ is true:

$P(1): 1 = \frac{1(1+1)}{2}$

i.e. $1 = 1$

$P(1)$ is true.

(this is often called our "base case")

(2) (Our "inductive step")

To show $P(n) \Rightarrow P(n+1)$ holds for all $n \geq 1$.

Let's take a direct approach:

So, we suppose/assume that $P(n)$ is true

i.e. $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Let's show that $P(n+1)$ is also true.

Let's take the L.H.S. of $P(n+1)$:

$$\underbrace{1 + 2 + \dots + n}_{\text{(note: this is the L.H.S. of } P(n))} + n+1$$

Thus

$$1 + 2 + \cdots + n + n+1$$

$$= \frac{n(n+1)}{2} + n+1 \quad \text{, as } P(n) \text{ is true}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

This is the RHS of $P(n+1)$,
i.e. $P(n+1)$ is true.

Hence $P(n)$ is true for all $n \geq 1$.

Eg. 2 Let's prove that
$7^n - 1$ is divisible by 6
for all $n \geq 1$.

Induction

1. Our $P(n)$ is $6 \mid 7^n - 1$

so $P(1)$ is $6 \mid 7^1 - 1 = 6$: true.

2. Assume $P(n)$ is true,
i.e. $6 \mid 7^n - 1$.

Let's show $P(n+1)$ is true,

i.e. $6 \mid 7^{n+1} - 1$.

Let's show $7^{n+1} - 1$ is a multiple of 6:

$$7^{n+1} - 1 = 7^n \cdot 7 - 1$$

$$= (7^n - 1) \cdot 7 + 6$$

As $P(n)$ is true, $7^n - 1 = 6k$ for some k

Hence

$$7^{n+1} - 1 = (7^n - 1) 7 + 6$$

$$= (6k) 7 + 6$$

$$= 6(7k) + 6$$

$$= 6(7k + 1).$$

Thus $P(n+1)$ is true.

Hence $P(n)$ is true for all $n \geq 1$.

Sets

For us, a set will mean a collection of objects, called elements.

We can list the elements and use curly brackets to show we're dealing with a set, eg. the set S might be $S = \{1, 2, 4\}$

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We use predicates to describe sets with infinitely many elements:

$$S = \{ 2n-1 \mid n \text{ is a positive integer} \}$$

→ This is the set of odd positive integers which is the set of all integers $2n-1$ <u>such that</u> n is a positive integer.

i.e. $S = \{ x \mid P(x) \}$ is the set of x <u>such that</u> $P(x)$ is true.

Some important sets:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers

$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \text{ is an integer and } q \text{ is a natural number} \right\}$

is the set of rational numbers

$\mathbb{R} = \{ \text{all decimal numbers} \}$

We have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$,

i.e. $\mathbb{N}$ is contained in $\mathbb{Z}$, etc.

<u>Also</u>, $\{\}$ or $\emptyset$ is the <u>empty set</u>, the set with no elements

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$\{\ \} = \{x \mid x \neq x\}$

- We say a set A is a <u>subset</u> of a set B if A is contained in B,

i.e. $\cancel{x} \in A \Rightarrow x \in B$.

if x is an element of A, then x is an element of B

When are two sets equal?
When they have the same elements.

So, we can show two sets A and B are equal by showing $A \subseteq B$ and $B \subseteq A$.

eg. let $A = \{n \mid n^2 \text{ is an odd integer}\}$
& $B = \{n \mid n \text{ is an odd integer}\}$.

Then $A = B$ (Why? look back)

- We'll see set operations next.