• REVIEW OF LAST CLASSES:

* Propositions → VACUOUS TRUTH
* Logical operators:
  - not(P), $\overline{P}$
  - P AND Q, $P \wedge Q$
  - P OR Q, $P \vee Q$
  - $P \Rightarrow Q$
  - $P \Leftrightarrow Q$

"Every dog in this room is red"

"The sun is blue $\Rightarrow$ 2 is prime" is TRUE.

We had just defined:

* LOGICAL EQUIVALENCE:

$P \Leftrightarrow Q$, or $P \equiv Q$

by the truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- In practice, two propositions are logically equivalent if they have the same columns in a truth table.

- EXAMPLE: let us show that

$$(P \Rightarrow Q) \Longleftrightarrow (\text{not } Q \Rightarrow \text{not } P)$$

P	Q	not P	not Q	P ⇒ Q	not Q ⇒ not P
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- $(\text{not } Q) \Rightarrow (\text{not } P)$ is called the contrapositive of $P \Rightarrow Q$

* DE MORGAN'S LAWS:

(i) $(\text{not}(P \wedge Q)) \equiv (\text{not } P \vee \text{not } Q)$

(ii) $(\text{not}(P \vee Q)) \equiv (\text{not } P \wedge \text{not } Q)$

proof of (i):

P	Q	not P	not Q	P∧Q	not(P∧Q)	not P ∨ not Q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

proof of (ii): as exercise.

* ~~LOGICALLY TRUE PROPOSITION!~~

- ~~Sometimes the truth value of a proposition may depend on a variable.~~

~~Example: – "Today it rains"~~

~~– "n is a prime number"~~

* LOGICALLY TRUE PROPOSITION

- A proposition that is true in every possible case is said to be logically true.

Example: P V not P is logically true.

P	not P	P V not P
T	T	T
T	F	T
F	T	T
F	F	T

- We can still afford to call these propositions true. We've already encountered logically true propositions before in the class. (Like De Morgan's laws).

* PROVING AN IMPLICATION IN PRACTICE

- For two given propositions P and Q, we will often want to prove that $P \Rightarrow Q$ is logically true.

- Why? | Then we know that if P is true, then Q is true.

Three strategies:

① DIRECT ARGUMENT

Assume that P is true, show that Q is true.

Example: For n an integer, n is ~~[illegible]~~ even $\Rightarrow n^2$ is even.

Suppose we can write $n = 2k$ for some integer k.

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is even. $\square$

② CONTRAPOSITIVE ARGUMENT.

Assume that Q is false, show that P is false.

Recall: $(P \Rightarrow Q) \equiv (\text{not } Q \Rightarrow \text{not } P)$

Example: n^2 even $\Rightarrow$ n even.

contrapositive: n odd $\Rightarrow$ n^2 odd.

Assume $n = 2k+1$ for some integer k.

then $n^2 = (2k+1)^2 = 4k^2 + 2(2k) + 1^2$

$= 2(2k^2 + 2k) + 1$

and n^2 is odd. $\square$

~~③ PROOF BY CONTRADICTION.~~

- ~~Assume P is true and Q is false, and derive a contradiction.~~
- ~~?~~

• PROOF BY CONTRADICTION :

- We've already said that for a proposition P, P is *either* TRUE or FALSE

- To prove P using a proof by contradiction, we assume P is FALSE, and deduce non(Q) for a proposition Q we know is true. since non (Q) and Q CANNOT be both TRUE, we have a CONTRADICTION we deduce that P is FALSE

~~EXAMPLE: To prove that $\sqrt{2}$ is an irrational number (i.e. that it cannot be written as a quotient of integers), we assume~~ $\cancel{\sqrt{2} = \frac{p}{q}}$ ~~and show~~

EXAMPLE: INFINITY OF PRIME NUMBERS

Assume there is a finite list

$p_1, \ldots, p_n$ of prime numbers.

Consider $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$.

- N is not divisible by any of the p_i, since it will leave a remainder of one upon division.
- either N is prime, and we have a contradiction
- either N is divisible by a prime number that is not in our list, and we have a contradiction.

→ SEE ALSO THE CLASSIC PROOF OF THE IRRATIONALITY OF $\sqrt{2}$ IF YOU ARE CURIOUS.

* PROOF BY INDUCTION:

- Sometime we consider propositions which truth value depends on a variable.

  example: " x is positive "

  " it rains on day d "

- In particular we may consider $P(n)$, where n is a natural integer.

- We often use a proof by induction to show that:

  For EVERY $n \in \mathbb{N}$, $P(n)$ is true.

EXAMPLE: For every $n \in \mathbb{N}$,

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

In order to do that we:

- **INITIALISE**: Show that $P(1)$ is TRUE.

- **PROVE THE INDUCTION STEP**:

  Show that

  $P(n)$ true for some $n \in \mathbb{N}$ $\Longrightarrow$ $P(n+1)$ TRUE.

REMARK ⚠: We assume $P(n)$ is true for some n and not for every n

(that would be what we want to prove!)