MS115

- CA1 : In-class test in Week 6
  → first hour of class

CA total : 25% → split between the 2 in-class tests

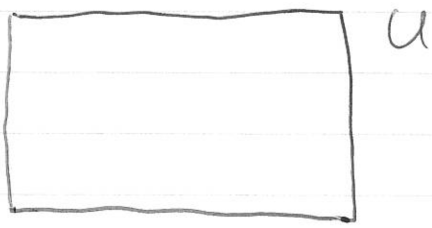
Examinable content : up to end of Week 4

Sets

Venn diagrams are a useful tool when dealing with sets.

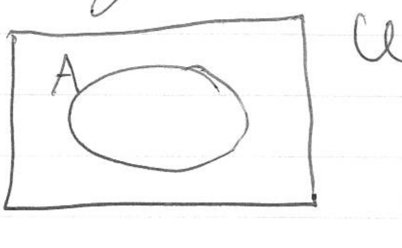
We start by considering a large set $\mathcal{U}$ that contains all the elements that we are interested in.

Eg. $\mathcal{U}$ might be the ("universal") set of all cars.
We use a rectangle to represent $\mathcal{U}$ in our Venn diagram:



Within $\mathcal{U}$, we might be interested in certain subsets, eg. $A = \{\text{Ford cars}\}$

We represent these subsets using circles within $\mathcal{U}$:



We recall that A is a subset of B, written $A \subseteq B$, if $x \in A \Rightarrow x \in B$.

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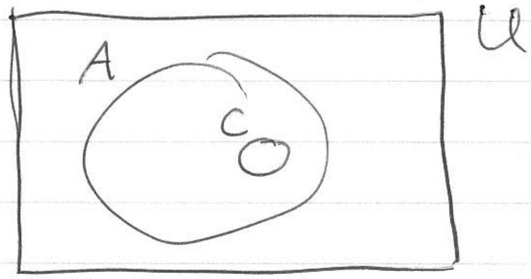
In the above, $A \subseteq \mathcal{U}$ as Ford cars are cars.

We might also be interested in subsets of A. For example, letting

$C = \{$ Ford cars built this year $\}$,

we have Venn diagram



We know that 2 sets A and B are equal exactly when

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

This is also clear from considering a Venn diagram.

What are the possible subsets of a set?

Let's take an easy example:

$$A = \{1, 2, 3\}$$

Let's list the subsets of A:

ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$,

$\{2, 3\}$, $\{1, 3\}$, $\{1, 2, 3\}$

(The "empty" or "null" set $\phi = \{\ \}$ is a subset of every set)

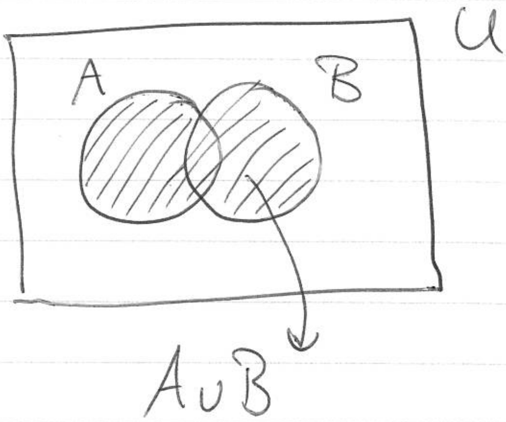
This example shows that a set with 3 elements has $2^3 = 8$ subsets.

We'll see that, in general, a set with n elements has 2^n subsets.

Set operations

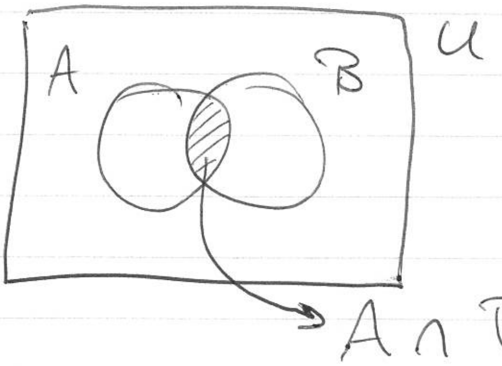
- The <u>union</u> of two sets A & B is the set $A \cup B = \{x \in U \mid x \in A$ **or** $x \in B\}$

<u>Venn diagram</u>



- The <u>intersection</u> of sets A & B is the set $A \cap B = \{x \in U \mid x \in A$ <u>and</u> $x \in B\}$

<u>Venn</u>



A and B are <u>disjoint</u> if $A \cap B = \phi$, i.e. A and B have no common elements.

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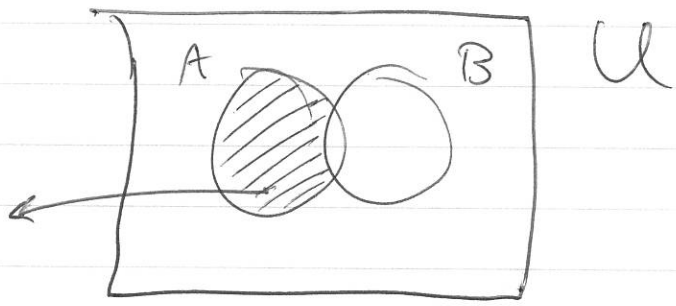
- We define the <u>complement of a set B relative to A</u> to be the set

$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$$

i.e. x is in A and x is not in B.

Venn



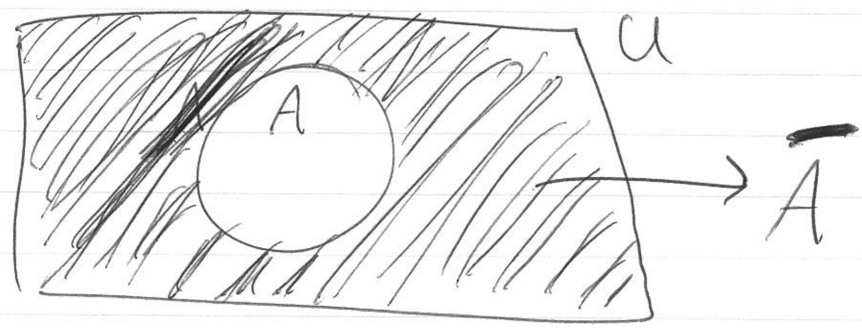
A-B

~~Exception~~

Note: $A - B = A$ if $A \cap B = \emptyset$

- As a particular case of the above definition, the <u>complement of a set A</u>, denoted $\overline{A}$, is the set $U - A = \{x \in U \mid x \notin A\}$.

Venn



<u>Examples</u>: Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 2, 3, 4, 5\}$.

Here $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\} = B \cup A$

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$A \cup C = \{1, 2, 3, 4, 5, 7\} = C \cup A$

$B \cap C = \{2, 4\} = C \cap B$

$C - A = \{2, 4\}$

$A - C = \{7\}$

If $U = \{1, 2, 3, \ldots, 9, 10\}$,

then $\bar{A} = U - A = \{2, 4, 6, 8, 9, 10\}$.

A and B are disjoint.

<u>Note</u>: Since subsets, unions, intersections and complements are defined using

$\Rightarrow$, or, and, not,

we can establish set identities using logical equivalences.

For example, we know from De Morgan's laws

that not (P or Q) $\equiv$ not P and not Q

Let P be the statement $x \in A$
& Q be the statement $x \in B$.

Then not $(x \in A$ or $x \in B) \equiv$ not $(x \in A)$ and not $(x \in B)$, which is the same as $\overline{A \cup B} = \bar{A} \cap \bar{B}$

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Similarly we have

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

on the basis of the other De Morgan law

(not (P and Q) $\equiv$ not P or not Q)

- We can show many such set identities on this basis

eg. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Use the corresponding logical equivalences

We can also get a sense of whether such identities hold using our Venn diagrams,

eg. consider the statement $A \cap (B \cup C)$
$= (A \cap B) \cup (A \cap C)$

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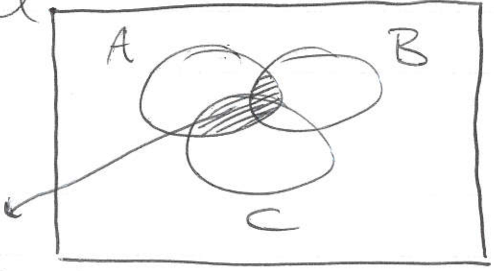
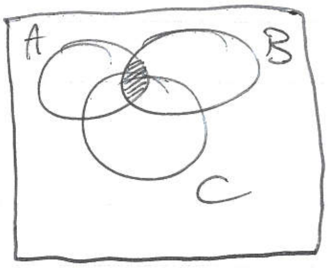
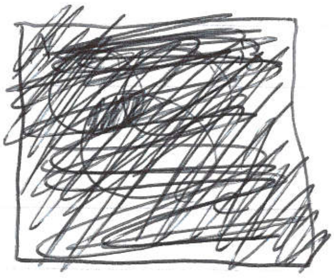
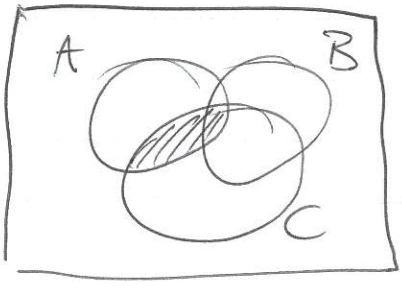


$A \cap (B \cup C)$







We're interested in counting the number of elements in a finite set (a set with finitely many elements).

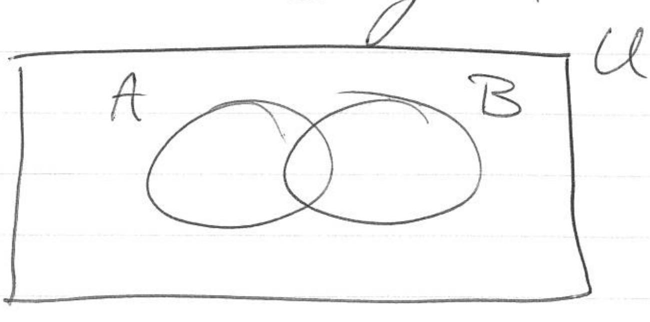
<u>Def$^{\underline{n}}$</u>: The <u>cardinality</u> of a finite set A is the number of elements in A, and is denoted by $|A|$.

<u>Eg.</u> For $A = \{\text{black, red, yellow}\}$ we have $|A| = 3$.

Inclusion - Exclusion allows us to count the elements in a union.

- If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

<u>Proof</u>: Using a Venn Diagram, we see that



$$|A \cup B| = |A \cap \overline{B}| + |A \cap B| + |B \cap \overline{A}|$$

(note: the union of the sets $A \cap \overline{B}$, $A \cap B$ and $B \cap \overline{A}$ gives A <u>and</u> these sets are disjoint)

Now $|A \cap \overline{B}| + |A \cap B| = |A|$
and $|B \cap \overline{A}| = |B| - |A \cap B|$

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Eg. Let's suppose 50 students in a course have a choice between 2 optional modules A and B.
Suppose 16 take A and 20 take B and 5 take both.
How many take neither?

Here $|U| = 50$, $|A| = 16$, $|B| = 20$ and $|A \cap B| = 5$.

Want $$|\overline{(A \cup B)}| = |U - (A \cup B)|$$

$$= |U| - |A \cup B|$$

As $$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 16 + 20 - 5 = 31,$$

we have $50 - 31 = 19$ students who take neither.

- Given 3 finite sets A, B and C, we have

$$|(A \cup B \cup C)| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

We can prove this in just the same way as the 2-set case above:

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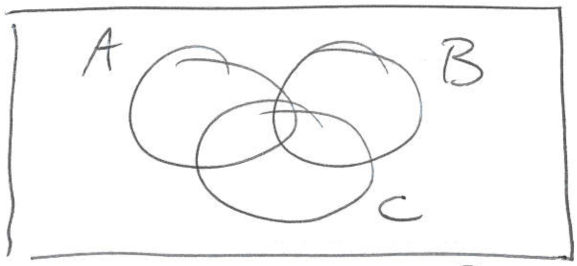
<u>Idea</u>: Express $A \cup B \cup C$ as a union of sets that are "pairwise disjoint" i.e. no pair has elements in common



~~Alternatively~~ Consider these sets

$A \cap B \cap C$, $A \cap B \cap \bar{C}$, $A \cap \bar{B} \cap C$, $A \cap \bar{B} \cap \bar{C}$, $\bar{A} \cap B \cap C$, $\bar{A} \cap B \cap \bar{C}$, $\bar{A} \cap \bar{B} \cap C$, $\bar{A} \cap \bar{B} \cap \bar{C}$.

Count their occurences in each of the sets on the r.h.s. of our expression,

$|A|$, $|B|$, $|C|$, $-|A \cap B|$, $-|A \cap C|$, $-|B \cap C|$, $+|A \cap B \cap C|$.

Not hard, but tedious : see handout.

- We can also consider the <u>Cartesian product</u>, $A \times B$ of two sets A and B.

$A \times B$ is the set of <u>ordered pairs</u> (a, b) where $a \in A, b \in B$.

<u>Eg.</u> Let $A = \{\text{red, yellow}\}$
and $B = \{1, 2, 3\}$,
then

$$A \times B = \{(\text{red},1), (\text{red},2), (\text{red},3), (\text{yellow},1), (\text{yellow},2), (\text{yellow},3)\}$$

Clearly $A \times B \neq B \times A$ in general, i.e. order matters.

Here $B \times A = \{(1,\text{red}), (1,\text{yellow}), \ldots, (3,\text{red}), (3,\text{yellow})\}$

Note that while $A \times B \neq B \times A$ here, they both have 6 elements.

For A & B finite sets, we have

$$|A \times B| = |A| \cdot |B|$$

- We can take the product of a finite set with itself n times:

$$A^n = \underbrace{A \times A \times \ldots \times A}_{n\ A\text{'s}}$$

$$= \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, \ldots, n\}$$

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Clearly, $|A^n| = |A|^n$

Eg. For $B = \{0, 1\}$

we have $B^8 = \{(a_1, \ldots, a_8) \mid a_i = 0 \text{ or } 1\}$

This is the set of bytes,

a set with $|B^8| = 2^8$ elements.