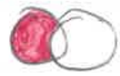
Thursday, october 10.

* REVIEW OF LAST CLASS:

CORRESPONDANCE WITH LOGICAL OPERATORS

→ Sets, elements.
→ VENN diagrams.
→ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ ← $\sqrt{2} \notin \mathbb{Q}$
→ Inclusion: $A \subset B$. $\Longrightarrow$
- * $\emptyset$ empty set
- * finding every subset of $\{1, 2, 3\}$
- * equality of set $\Longleftrightarrow$

→ Set operators:
- * UNION $A \cup B$  OR
- * INTERSECTION $A \cap B$  AND
- * COMPLEMENT relative to A: $A - B$ 
- * COMPLEMENT $\bar{A} = U - A$  NOT

→ We've finished proving a "De Morgan's law" for sets:

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

→ We can show many set identities like this:

e.g.: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

→ Use VENN diagram to convince yourself.

→ Proof using the definitions.

* CARDINALITY:

We are interested in counting the number of elements in a set, when this number is finite.

Vocabulary: we say that a set with a finite number of elements is a

FINITE SET

DEFINITION: The CARDINALITY of a finite set A is the number of elements in A, and is denoted by $|A|$.

Ex: $A = \{\text{black}, \text{red}, \text{yellow}\}$.

$|A| = 3$.

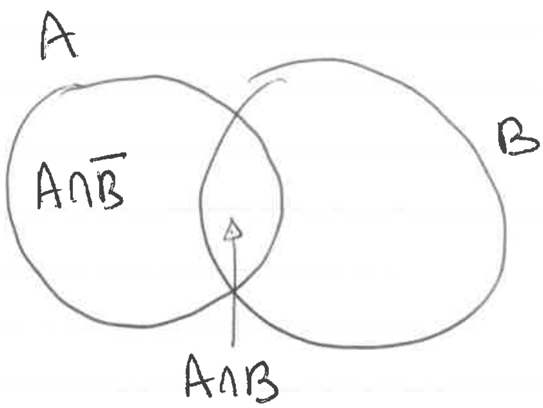
REMARK: $|\mathbb{N}|$ is not well defined as we have an infinite number of elements in $\mathbb{N}$!

* COMPUTING CARDINALS

* A SIMPLE SITUATION

- We start with two sets A and B, that are finite.

- Notice that $A = (A \cap B) \cup (A \cap \overline{B})$



- Moreover, $A \cap \overline{B}$ and $A \cap B$ are disjoints.

- Thus we can write

$$|A| = |A \cap B| + |A \cap \overline{B}| .$$

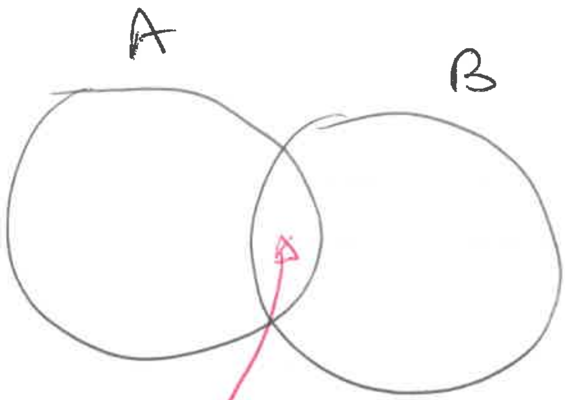
→ Expressing sets as disjoint unions allow us to add CARDINALS.

→ Similarly of course: $|B| = |B \cap A| + |B \cap \overline{A}|$.

* CARDINAL OF THE UNION

Let A and B be finite sets. We have

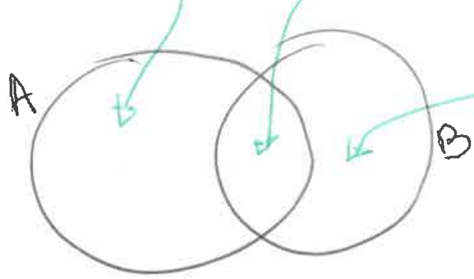
$$|A \cup B| = |A| + |B| - |A \cap B|$$



The idea is that we count $|A \cap B|$ twice if we just write $|A| + |B|$, so we remove it once.

PROOF: We express $A \cup B$ as a disjoint union (we "cut it into pieces")

$$A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (B \cap \bar{A})$$



So we have $|A \cup B| = |A \cap \bar{B}| + |A \cap B| + |B \cap \bar{A}|$ (*)

- we then recognize

$$|A \cap \bar{B}| + |A \cap B| = |A| \quad ①$$

and we use $|B| = |B \cap A| + |B \cap \bar{A}|$

to obtain $|B \cap \bar{A}| = |B| - |B \cap A|$ ②

Using ① and ② in ⊛ we have

$$\boxed{|A \cup B| = |A| + |B| - |B \cap A|}$$ and our job is done.

REMARK: We will also use

$$|A \cap B| = |A| + |B| - |A \cup B|$$ sometimes.

(It is a simple consequence of our result).

APPLICATION: Let us suppose 50 students in a course have a choice between two optional modules, A and B.

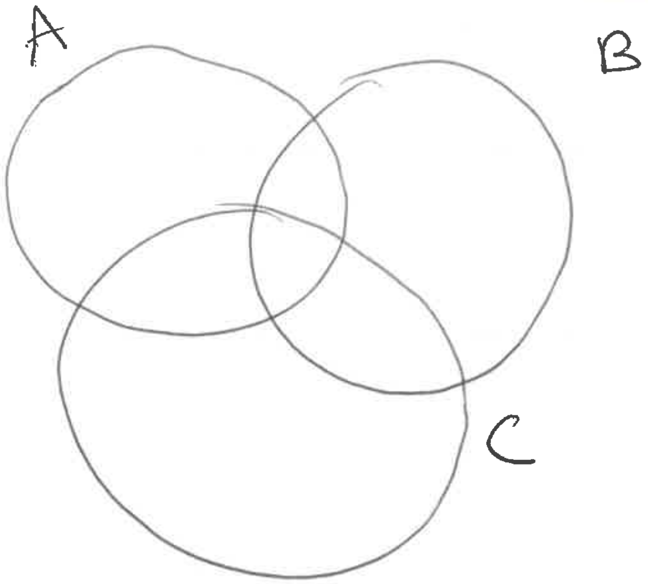
Suppose 16 take A and 20 take B, 5 take both.

How many take neither?

* CARDINAL OF THE UNION OF THREE SETS:

For three finite sets A, B, C we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Idea of the proof: Express $A \cup B \cup C$ as a union of sets that are "pairwise disjoints".

* CARTESIAN PRODUCT AND CARDINALITY

DEFINITION: Given two sets A and B, the set of ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the cartesian product of A and B, denoted by $A \times B$.

Ex: $A = \{\text{red}, \text{yellow}\}$

$B = \{1, 2, 3\}$.

$$A \times B = \{(\text{red}, 1), (\text{red}, 2), (\text{red}, 3), (\text{yellow}, 1), (\text{yellow}, 2), (\text{yellow}, 3)\}.$$

REMARK. $(\text{red}, 1) \neq (1, \text{red})$ since we consider ordered pairs.

as a consequence $A \times B \neq B \times A$.

$B \times A = \{(1, \text{red}), (2, \text{red}) \ldots$

CARDINAL: If A and B are finite,

$$|A \times B| = |A| \cdot |B|$$

REMARK: • We can consider

$$\underbrace{A \times \cdots \times A}_{n \text{ times}} = A^n.$$

If A is finite we have $|A^n| = |A|^n$.

• You already know $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ which is the plane with coordinates.

$$\mathbb{R}^2 = \left\{ (x, y) \text{ s.t } x \in \mathbb{R} \text{ and } y \in \mathbb{R} \right\}.$$