MS115

- **Recall**: The Cartesian product of two sets A and B is the set of ordered pairs

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

Eg. $A = \{1, 2\}$ and $B = \{x, y, z\}$, then

$$A \times B = \{ (1,x), (1,y), (1,z), (2,x), (2,y), (2,z) \}$$

$$|A \times B| = |A|\,|B|$$

Def^{n} A binary relation between two sets A and B is a subset R of $A \times B$.

In the case where $A = B$, we refer to R as being a relation on A.

- Taking the above example of A and B, we can choose

$$R = \{ (1,x), (1,z), (2,y) \} \text{ for example.}$$

This tells us that
- 1 is related to x by R
- 1 is related to z by R
- 2 is related to y by R

We write $1Rx$, [redacted] Rz and $2Ry$ to denote these relationships.

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- Certain relations on a set A are particularly important.

Equivalence relations

Defⁿ A relation R on a set A is said to be an equivalence relation if it has the following three properties

- (reflexive) if $x \in A$, then $x R x$
- (symmetric) if $x, y \in A$ such that $x R y$, then $y R x$
- (transitive) if $x, y, z \in A$ such that $x R y$ and $y R z$, then $x R z$

Eg.
- For $A = \mathbb{Z}$, the relation R defined by "is less than", is transitive:

  if $x < y$ and $y < z$, then $x < z$;

  but is not symmetric or reflexive.

- For $A = \mathbb{Z}$, the relation R defined by "is less than or equal to" is

③

transitive: as if

$x \leq y$ and $y \leq z$, then $x \leq z$;

but is not symmetric:

if $x \leq y$ then $y \nleq x$ in general.

Not symmetric:

we don't have

$x \leq y \Rightarrow y \leq x$ for all $x, y \in \mathbb{Z}$

eg. $2 \leq 3$ but $3 \nleq 2$

It is reflexive however:

$x \leq x$ for all $x \in \mathbb{Z}$

- <u>Eg.</u> For $A = \mathbb{Z}$ and R is defined by "is equal to", R **is** an equivalence relation on A.

Clear: reflexive ✓ $x = x$ for all $x \in \mathbb{Z}$

symmetric ✓ if $x = y$ then $y = x$ for all $x, y \in \mathbb{Z}$

transitive ✓ if $x = y$ and $y = z$, then $x = z$ for all $x, y, z \in \mathbb{Z}$

- <u>Eg.</u> For $A = \mathbb{Z}$, let R be the relation on A given by

xRy exactly when $y - x$ is even.

(4) Let's show this is an equivalence relation:

<u>reflexive</u>: for $x \in \mathbb{Z}$, we have

$$x - x = 0 = 2(0),$$

whereby $x R x$.

<u>symmetric</u>: for $x, y \in \mathbb{Z}$, suppose we have $x R y$. Then $y - x = 2k$ for some $k \in \mathbb{Z}$

Therefore, $x - y = 2(-k)$,

whereby $y R x$.

<u>transitive</u>: for $x, y, z \in \mathbb{Z}$, suppose we have

$x R y$ and $y R z$.

Thus we have

$y - x = 2k$ and $z - y = 2l$

for some $k, l \in \mathbb{Z}$

Thus $(z - y) + (y - x) = 2l + 2k$

i.e. $z - x = 2(l + k)$

i.e. $x R z$.

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Loosely speaking, an equivalence relation on a set A is a relation that ties together elements that share a common property.

Eg. Let A be the set of all people with hair and consider

$$R = \{(x,y) \in A \times A \mid x \text{ and } y \text{ have the same hair colour}\}$$

R is clearly an equivalence relation on A

Moreover, R appears to divide A into pairwise disjoint subsets,

eg. {people with blond hair},
{people with black hair},
{people with brown hair}, etc.

We'll make this precise.

Def^n: A partition of a set A is a collection of non-empty subsets $A_1, \dots, A_n$ of A satisfying

(i) $A = A_1 \cup A_2 \cup \dots \cup A_n$

and (ii) $A_i \cap A_j = \phi$ for $i \neq j$

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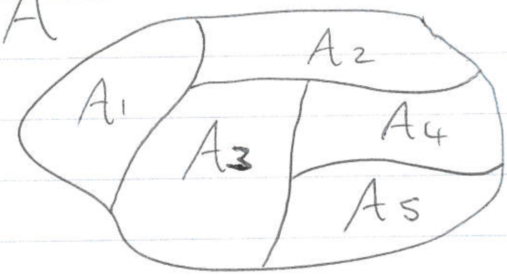
This notion, which we'll revisit, makes counting easier:

eg. A



We have $|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$ by the inclusion-exclusion principle

We'll show that equivalence relations define partitions.

Let R be an equivalence relation on set A.

Def^n: The equivalence class of any $x \in A$ is the set

$$E_x = \{ y \in A \mid y R x \}$$

Theorem: Let R be an equivalence relation on A, a non-empty set. Then, the distinct equivalence classes form a partition of A.

Proof: (i) We show the equivalence classes are non-empty subsets of A.

By definition, E_x is a subset of A. Also, $x \in E_x$ since R is reflexive. Thus, E_x is non-empty.

⑦

(ii) As E_x is a subset of A, the union of the equivalence classes is also a subset of A.

Also, if $x \in A$, then $x \in E_x$, so A is a subset of the union of the equivalence classes.

(iii) Finally, we must show that distinct equivalence classes are disjoint.

We'll use the contrapositive argument: we'll show that non-disjoint equivalence classes are not distinct. (i.e. the same).

Suppose $E_x \cap E_y \neq \phi$.

Let $z \in E_x \cap E_y$.

Then $z R x$ and $z R y$.

Hence $x R z$ by symmetry.

Hence $\underline{\underline{x R y}}$ as $x R z$ and $z R y$ (transitivity).

Now, given that $x R y$, we'll show that $E_x = E_y$.

To show equality of these sets (in the usual way)

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let $z \in E_x$.

Hence $z R x$.

As we know that $x R y$, we get $z R y$ by transitivity.

Hence $z \in E_y$.

Thus $E_x \subseteq E_y$.

We can show that $E_y \subseteq E_x$ in the same way.

- Another important class of relations, on sets A and B, are <u>functions</u>.

<u>Def^n</u>. A function from a set A to a set B is a relation between A and B which satisfies two properties:

1. every element in A is related to some element in B.

and 2. no element in A is related to more than one element in B.

<u>i.e.</u> given $a \in A$, there is exactly one $b \in B$ such that a is related to b by our function.

We usually write
$$f : A \to B$$ and

write $b = f(a)$ for $a \in A$,

and call b the image of a under f.

The set A is called the <u>domain</u> of the function.

The set B is the <u>codomain</u> of the function.

<u>Eg</u>. Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$.

The relation
$$R_1 = \{(a, x), (b, y), (b, z)\}$$
is not a function as b is not related to exactly one element in B.

The relation
$$R_2 = \{(a, x), (b, x), (c, z)\}$$
is a function.

- The <u>range</u> of a function f is the set of all images of elements of A under f.

eg. $\text{Range}(R_2) = \{x, z\}$.

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Some functions are "invertible":

If R is a relation on sets A and B,

$$R = \{(a,b) \mid a \in A, b \in B\},$$

then the inverse relation is the relation on B and A given by

$$R^{-1} = \{(b,a) \mid b \in B, a \in A\}$$

We say a function $f : A \to B$ is <u>invertible</u> if its inverse relation f^{-1} is a function $f^{-1} : B \to A$.

With some thought, we see that a function $f : A \to B$ is invertible if

every $b \in B$ is the image of exactly one $a \in A$.

- We can also "compose" functions:
- If R is a binary relation between A and B and S is a binary relation between B and C, then the <u>composition of R and S</u>, written $S \circ R$ ("S after R") is a binary relation between A and C given by

$$S \circ R = \{(a,c) \in A \times C \mid aRb \text{ and } bSc \text{ for some } b \in B\}$$

If $f : A \to B$ and $g : B \to C$ are functions, then the composition of f and g, written $g \circ f$ is the <u>function</u>

$$g \circ f : A \to C$$

given by $g \circ f(a) = g(f(a))$