(1) MS 115

Recall: When the composition $g \circ f$ of two functions f and g is defined, we evaluate it by "applying g after f",

i.e.
$$g \circ f(x) = g(f(x))$$

- Similarly, the composition $f \circ g$ is evaluated by "applying f after g",

i.e. $$f \circ g(x) = f(g(x))$$

Eg: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x+1$

and

$g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$

Then $$g \circ f(x) = g(f(x)) = g(2x+1)$$
$$= (2x+1)^2 = 4x^2+4x+1,$$

whereas $$f \circ g(x) = f(g(x)) = f(x^2)$$
$$= 2x^2+1.$$

Note: $f \circ g$ and $g \circ f$ are generally different functions, as above.

Recall: A function $f : A \to B$ is invertible if its inverse relation $\{(f(a), a) \mid f(a) \in B, a \in A\}$ is a function from B to A.

We write f^{-1} to denote this inverse function.

If $f: A \to B$ has an inverse $f^{-1}: B \to A$, then we can compose $f \circ f^{-1}$ and $f^{-1} \circ f$. As you might expect, these compositions leave the elements of their domain sets unchanged.

<u>Eg.</u> $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x+1$ has inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = x-1$.

We see that

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = f(x-1)$$

$$= (x-1)+1 = x$$

and $f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(x+1)$

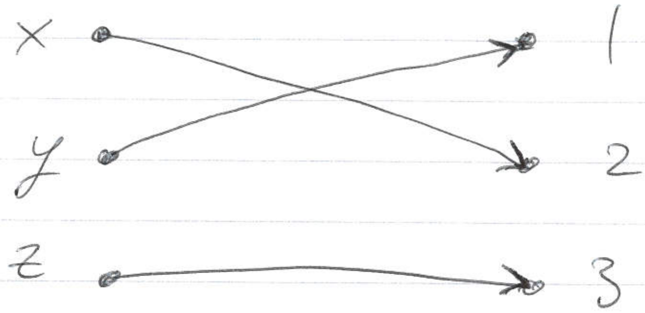
$$= (x+1) - 1 = x.$$

- As we've seen, we can determine whether a function $f: A \to B$ is invertible in the case where A and B are finite sets by looking at the relation with the aid of a picture:

Eg. For $A = \{x, y, z\}$ and $B = \{1, 2, 3\}$, and the function $f : A \to B$ given by

$$f(x) = 2, \quad f(y) = 1 \text{ and } f(z) = 3,$$

we consider the picture

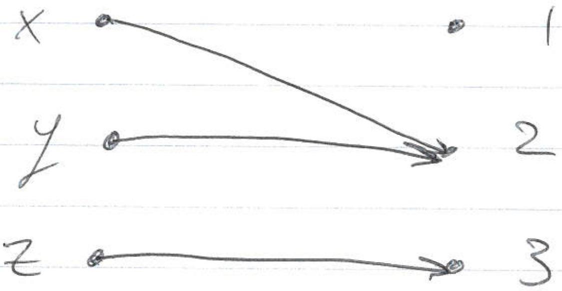


Here, we recognise that <u>every element of B</u> is the image of <u>exactly one</u> $a \in A$, whereby f is invertible in this case.

By contrast, the function $g : A \to B$ given by

$$g(x) = 2, \quad g(y) = 2 \text{ and } g(z) = 3$$

is not invertible:



For f as above, the inverse function is $f^{-1} : B \to A$ given by

$$f^{-1}(1) = y, \quad f^{-1}(2) = x \text{ and } f^{-1}(3) = z.$$

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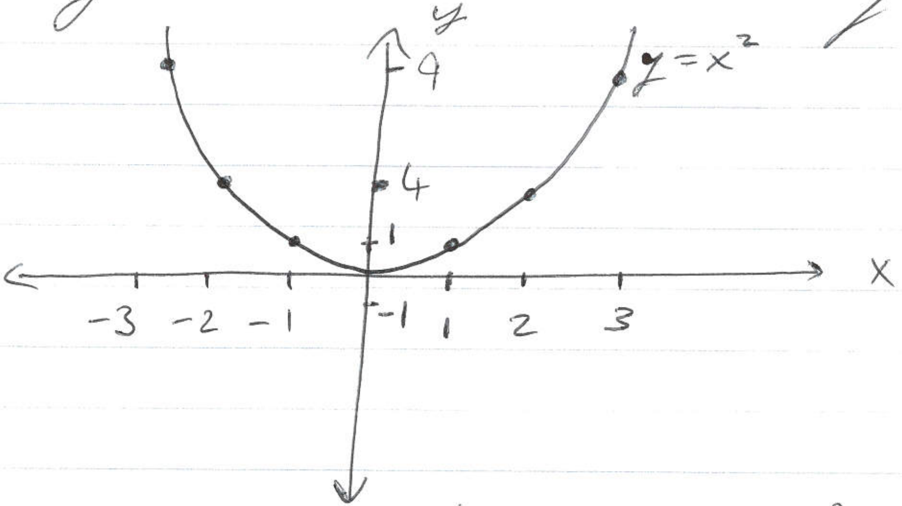
In the case where A and B are infinite sets, we must write $f: A \to B$ in terms of a formula, but again can determine whether f is invertible by looking at a picture:

the graph of a function $f: \mathbb{R} \to \mathbb{R}$

is the curve consisting of points (x, y) in $\mathbb{R} \times \mathbb{R}$ such that $y = f(x)$.

Eg. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ has graph



We recognise that this function is not invertible as we don't have that every element of $\mathbb{R}$ is the image of exactly one element of $\mathbb{R}$.

Eg. • no negative number is the image of any $x \in \mathbb{R}$

• each positive number is the image of 2 x values

eg. 4 is the image of -2 and $+2$.

To determine the inverse of an invertible function f, we express the input of f in terms of its output:

__Eg.__ For $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 2$, we write $y = 3x + 2$ and express x in terms of y:

$$y = 3x + 2 \implies y - 2 = 3x$$

$$\implies \frac{y-2}{3} = x$$

i.e. $x = \frac{1}{3}(y-2)$

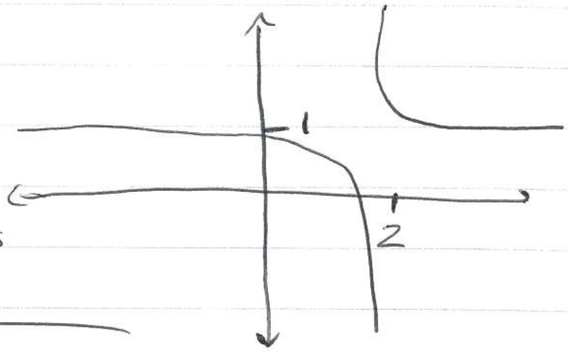
Thus, the function $g(y) = \frac{1}{3}(y-2)$ is the inverse of f, as when we input the value y into g we get x as its output!

__Exercise:__ Show $g \circ f(x) = x$ and $f \circ g(y) = y$.

__Eg.__ The function f given by $f(x) = \dfrac{x+1}{x-2}$ has domain $\mathbb{R} \setminus \{2\}$ (i.e. $f(x)$ is defined when $x \neq 2$)

and it can be shown to have codomain $\mathbb{R} - \{1\}$.

Aside: Graph would look something like this



It is invertible and its inverse can be found as above (i.e. by expressing the input in terms of the output):

$$y = \frac{x+1}{x-2} \Rightarrow y(x-2) = x+1$$

$$\Rightarrow yx - 2y = x + 1$$

$$\Rightarrow yx - x = 2y + 1$$

$$\Rightarrow x(y-1) = 2y+1$$

$$\Rightarrow x = \frac{2y+1}{y-1}$$

Thus the function $g(y) = \frac{2y+1}{y-1}$ is the inverse of $f(x) = \frac{x+1}{x-2}$.

- We'll consider linear functions $y = ax + b$, $a, b \in \mathbb{R}$.

These functions, as their name suggests, have straight-line graphs.

The straight line thus associated with the function $y = ax + b$ must therefore be described by the values a and b.

The value a is the change in y brought about by a one-unit increase in x,

e.g. for $y = 6x - 13$,

we see that y increases by 6 when x increases by 1.

This number a defines the <u>slope</u> of the line.

Also, the value b is the value of y when $x = 0$ (as $y = ax + b$).

This is called the <u>y-intercept</u> of the line (i.e. the y-value where the line crosses the y-axis).

Thus, making a choice of a and b determines the straight-line graph of $y = ax + b$.

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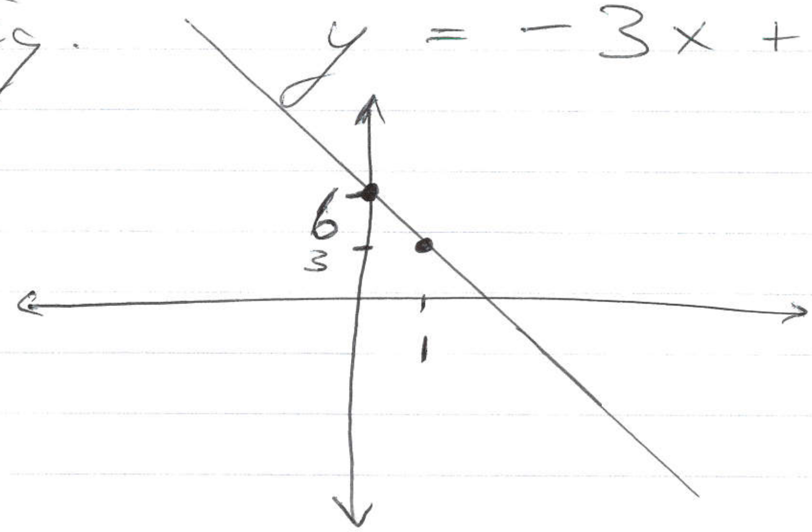
Eg. $y = -3x + 6$



Vice-versa, given two points on a line, we can recover the values a and b.

eg. if (x_1, y_1) and (x_2, y_2) are 2 points on the straight-line graph of $y = ax + b$,

then $y_1 = ax_1 + b$
and $y_2 = ax_2 + b$,

so $y_2 - y_1 = ax_2 + b - (ax_1 + b)$

$\Rightarrow \quad y_2 - y_1 = a(x_2 - x_1)$

$\Rightarrow \quad a = \dfrac{y_2 - y_1}{x_2 - x_1}.$

- The straight-line graphs of 2 linear functions $f(x) = ax + b$ and $g(x) = cx + d$ will intersect in one point if the lines have different slopes, i.e. $a \neq c$

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We can easily find the point of intersection of such functions:

eg. for $f(x) = 2x+3$ and
$g(x) = x-4$,

we solve for $y = 2x+3$
and $y = x-4$.

This gives $2x+3 = x-4$

$\Rightarrow \quad x = -7$

$\Rightarrow \quad y = x-4 = (-7)-4 = -11$

Thus $(x,y) = (-7,-11)$ is the point of intersection.