(1)

MS115

- CA2: In-class test in Week 10
  → details to follow
- <u>Recall</u>: Given a demand function $Q_D = -2P + 10$ and a cost function $TC = 2Q_D + 4$, we can express total revenue $TR = P \times Q_D$ and profit $\Pi = TR - TC$ as quadratic functions in Q_D.

To do this, we invert our demand function:

$$Q_D = -2P + 10 \Rightarrow 2P = -Q_D + 10$$

$$\Rightarrow \quad P = \left(-\tfrac{1}{2}\right)Q_D + 5.$$

Hence, $TR = P \times Q_D = \left(\left(-\tfrac{1}{2}\right)Q_D + 5\right)Q_D$

$$= -\frac{Q_D^2}{2} + 5Q_D.$$

Hence, our profit function is

$$\Pi = TR - TC = -\frac{Q_D^2}{2} + 5Q_D - (2Q_D + 4),$$

i.e. $$\Pi = -\frac{Q_D^2}{2} + 3Q_D - 4.$$

2

We can sketch the graph of π by determining its vertical intercept (i.e. π when $Q_D = 0$), and its horizontal intercepts (i.e. Q_D when $\pi = 0$).

For $\pi = -\frac{Q_D^2}{2} + 3Q_D - 4$, we have vertical intercept $\pi(0) = -4$.

We find the horizontal intercepts by solving for Q_D such that

$$-\frac{Q_D^2}{2} + 3Q_D - 4 = 0.$$

We can do this using the $-b$ formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $-\frac{Q_D^2}{2} + 3Q_D - 4 = 0$,

we have $a = -\frac{1}{2}$, $b = 3$ and $c = -4$,

whereby $Q_D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(-\frac{1}{2})(-4)}}{2(-\frac{1}{2})}$

$$\Rightarrow Q_D = \frac{-3 \pm \sqrt{9 - 8}}{-1}.$$

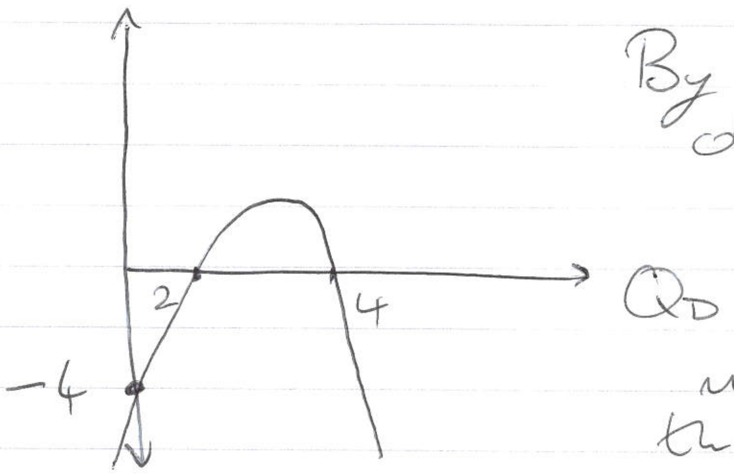
Thus we have horizontal intercepts

$Q_D = +3 - 1 = 2$ and $Q_D = 3 + 1 = 4$.

3

The graph of our graph function is the $\cap$-shaped curve that passes through our 3 intercepts:

eg. for $\pi = -\frac{Q_D^2}{2} + 3Q_D - 4$, we have



By the symmetry of the curve, the maximum value of π occurs at the midpoint between the horizontal intercepts.

The midpoint between

$$-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$$

is clearly $-\frac{b}{2a}$.

Here, our profit function has its maximum value at

$$Q_D = -\frac{b}{2a} = -\frac{3}{2(-\frac{1}{2})} = 3$$

(Alternatively, π has its max value at the midpoint between 2 and 4, which is

$$Q_D = \frac{2+4}{2} = \frac{6}{2} = 3 \text{)}$$

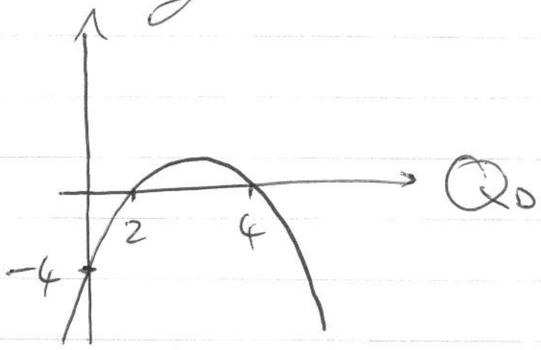
The max. profit is thus given by

$$\pi(3) = -\frac{(3)^2}{2} + 3(3) - 4$$

$$= -\frac{9}{2} + 9 - 4 = \frac{1}{2}$$

4

Looking at our graph of π,



we see that we have negative profit (loss) when

$0 \leq Q_D < 2$ and $Q_D > 4$;

we have zero profit (i.e. we break even) when $Q_D = 2$ and $Q_D = 4$;

and we have positive profit when

$2 < Q_D < 4$.

5

Counting and Combinatorics

We seek systematic methods of counting the number of ways certain events can occur.

We start with an obvious observation:

Addition Principle of Counting

If an event E_1 can occur in n ways and an event E_2 can occur in m ways <u>and</u> E_1 and E_2 cannot occur at the same time, then the event E_1 or E_2 can occur in $n+m$ ways.

Eg. If I can travel to DCU in 3 ways (by car, bus or bike), and I can travel to Liverpool in 2 ways (by plane or ferry), then I can travel to DCU or Liverpool in 5 ways.

As we'll see when we discuss probability, events can be viewed as sets of outcomes. Events E_1 and E_2 are <u>mutually exclusive</u> if the event E_1 and E_2 cannot occur (i.e. E_1 and E_2 cannot occur at the same time). Viewing E_1 and E_2 as sets of outcomes, we have that mutually exclusive events E_1 and E_2 are disjoint, i.e. $E_1 \cap E_2 = \emptyset$.

Thus, the Addition Principle that

$$|E_1 \cup E_2| = |E_1| + |E_2|$$

for E_1 and E_2 mutually exclusive follows from Inclusion - Exclusion (recall: $|A \cup B| = |A| + |B| - |A \cap B|$).

More generally, if E is the compound event that E_1 or E_2 or $\ldots$ or E_k occurs, and no two of the events $E_1, \ldots, E_k$ can occur at the same time, then the set E is partitioned by the sets $E_1, \ldots, E_k$, whereby

$$|E| = |E_1| + \ldots + |E_k|$$

by Inclusion - Exclusion,

i.e. E can occur in $n_1 + n_2 + \ldots + n_k$ ways, where E_i can occur in n_i ways.

Perhaps more importantly, we also have:

<u>Product Principle of Counting</u>: If an event E_1 can occur in n ways and an event E_2 can occur in m ways, then <u>E_1 followed by E_2</u> can occur in $nm = (n)(m)$ ways

More generally, if E_i can occur in n_i ways, then E_1 followed by E_2 followed by $E_3 \ldots$

followed by E_k can occur in $n_1 n_2 \dots n_k$ ways.

Viewing events as sets of outcomes, this follows from the fact that a sequence of outcomes is an element of $E_1 \times E_2 \times \dots \times E_k$ (the Cartesian product of $E_1, \dots, E_k$), and that

$$|E_1 \times E_2 \times \dots \times E_k| = |E_1| . |E_2| \dots |E_k|.$$

Eg. A Hungarian licence plate consists of 3 letters followed by 3 digits. Letting E_1 be the choice of first letter, E_2 the choice of second letter, ..., E_6 the choice of the third digit, we have that there are

$$(26)(26)(26)(10)(10)(10)$$

possible plates.

Eg. Suppose a compound proposition P is made up of the simple propositions $P_1, \dots, P_n$. Letting E_1 be the choice of truth value for P_1, E_2 be the choice of truth value for P_2, etc., we see that there are

$$\underbrace{(2)(2)\dots(2)}_{n \text{ times}} = 2^n$$

rows in the truth table of P.

Eg. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set with n elements, i.e. $|S| = n$. Consider a subset T of S.

Letting E_1 be the choice of whether x_1 belongs to T, E_2 be the choice of whether x_2 belongs to T, etc., we see that there are

$$\underbrace{(2)(2)\cdots(2)}_{n \text{ times}} = 2^n$$

possible subsets T of S.

Eg. For $S = \{x_1, x_2, x_3\}$, we have $2^3 = 8$ subsets:

$$\phi, \{x_1\}, \{x_2\}, \{x_3\},$$
$$\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, S.$$

The main counting problem we'll consider is the following:

In how many ways can k objects be selected from n objects?

We will consider this question in 4 different scenarios, depending on whether or not the order of selection is important and whether or not objects can be repeatedly selected.

Case 1 Order of selection is important and repeated selection is allowed.

As order of selection is important, one object is selected first, one second, and so on until all k objects are selected.

As repeated selection is allowed (i.e. we're selecting and then replacing) we have n choices for the first object, and also n choices for the second object, and each subsequent object.

Hence, by the Product Principle, we have

n^k possible choices.

Eg. The roles of President, followed by Treasurer and finally Secretary are being assigned in a committee of 5 people. If it is possible for a person to hold up to 3 roles (i.e. repeated selection is allowed), then there are $5^3 = 125$ possible outcomes to this process.

Case 2: Order of selection is important and repeated selection is not allowed

Again, as order is important, one object is selected first, one second and so on until k objects are selected. As repeated selection is <u>not</u> allowed, we have n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, $\ldots$, $n-(k-1)$ choices for the k^{th} object

Hence, by the Product Principle, we have

$$n(n-1)(n-2)\cdots(n-k+1)$$

possible choices.

Eg. Consider our previous example where the roles of President, Treasurer and Secretary are assigned in a committee of 5 people. Suppose now that nobody can hold more than one role. Then we have

$$(5)(4)(3) = 60$$

possible outcomes to this selection process.