1

CA2 : Week 10 (14 days from now)

Examinable material :

Weeks **5-8** inclusive
Problem Sheets 5 - 8 inclusive

- As products of consecutive positive integers in decreasing order play an important role in counting problems, they have their own notation.

Def$^{\underline{n}}$: If n is a positive integer, we define n factorial, written $n!$, to be the product of the first n positive integers,

i.e. $n! = n(n-1)(n-2)\ldots(2)(1)$

Eg. $5! = 5(4)(3)(2)(1) = 120$

Moreover, for $n=0$, we define $0! = 1$.

(this is just a useful convention).

- As above, if order is important and repeated selection is not allowed, we can select k objects from n in

$(n)(n-1)(n-2)\cdots(n-k+1)$.

We can express this number as

$$\frac{n!}{(n-k)!}, \text{ since } \frac{n!}{(n-k)!} = \frac{n(n-1)\cdots\cancel{(n-k)}\cdots\cancel{(1)}}{\cancel{(n-k)}\cdots\cancel{(1)}}$$

$$= n(n-1)\cdots(n-k+1).$$

[Eg. The number of ways 3 committee roles can be assigned to 5 candidates if nobody can hold more than one role,

is $\frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5(4)(3)\cancel{(2)}\cancel{(1)}}{\cancel{(2)}\cancel{(1)}}$

$= 60$ ways.]

- An important special case of the above occurs when $\underline{\underline{n=k}}$.

Here we are choosing n objects from n <u>in order</u> but <u>without repeated selection</u>.

This corresponds to ordering the n objects,
i.e. arranging them in some order.

Our formula says that this process of ordering (or "permuting", or "arranging") n objects can be done in $\frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$

Again, as above, this follows from the Product Principle, as we have n choices for the first object, $n-1$ choices for the second, ..., 1 choice for the $n^{\underline{th}}$ object.

Eg. The letters a, b, c can be ordered in $3! = 6$ ways. They are abc, acb, bac, bca, cab, cba.

Eg. Six people can be lined up in $6! = 720$ ways.

- We next treat the cases of selecting k objects from n when order is not important.

Case 3 Order not important and repeated selection is not allowed.

Here, we seek to make an unordered selection of k distinct objects from n objects.

We can think of this as choosing

a subset of size k from a set of size n (as the ordering of elements is not important in a set).

The number of ways of doing this is called "n choose k" and is written as

$\binom{n}{k}$ (or sometimes "$C(n,k)$" or "${}^nC_k$")

We have that $\boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$

__Why?__ We note that we can make an ordered selection of k distinct objects from n by first making an unordered selection of k distinct objects and then ordering them.

__Eg.__ To make an ordered selection of $k=3$ objects from $\{a,b,c,d,e\}$, we can make an unordered selection ($\{a,c,e\}$ say) and then order it (giving one of ace, aec, cae, cea, eac, eca).

Thus, by the Product Principle, the number of ways of making an ordered sel$\underline{n}$ of k distinct objects is the number of ways of making an unordered sel$\underline{n}$

times the number of ways of ordering them,

i.e. $\frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$,

whereby $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Eg. The number of ways a committee of 4 can be chosen from 10 people is

$$\binom{10}{4} = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!}$$

$$= \frac{10(9)(8)(7)}{4(3)(2)(1)} = \frac{5040}{24} = \underline{210}$$

Eg. Suppose we wish to form a committee of 6 from a panel of 5 males and 6 females.

(i) In how many ways can this be done?

(ii) In how many ways can it be done if we must have an equal number of males and females?

(i) Here, we're choosing 6 people from 11, whereby we have

$$\binom{11}{6} = \frac{11!}{5!\,6!} = \frac{11(10)(9)(8)(7)}{(5)(4)(3)(2)(1)}$$

$$= 462$$

possible committee choices.

(ii) We must choose 3 males and 3 females. We have $\binom{5}{3}$ possible choices of 3 males and $\binom{6}{3}$ possible choices of 3 females.

Hence, by the Product Principle, we have $\binom{5}{3} \times \binom{6}{3}$ ways of forming a committee with equal male-female representation,

i.e. we have $\binom{5}{3} = 10 \times \binom{6}{3} = 20 = 200$ possible committees.

<u>Case 4</u>: Order of selection is <u>not important</u>, and repeated selection <u>is</u> allowed.

Let's consider an example, where we choose $k = 5$ objects from $n = 3$ distinct objects a, b, c with repeated selection allowed.

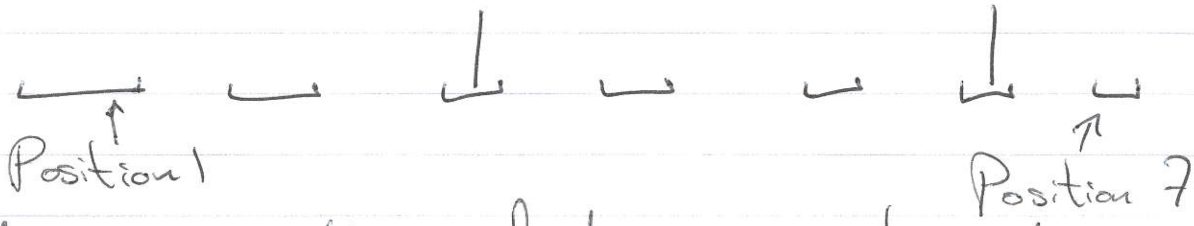
Thus, our selection might be a, a, b, b, c or a, b, b, b, c, and we consider the selections a, a, a, b, c and a, b, c, a, a to be the same, as order of selection is not important.

Grouping the a's first, the b's second and the c's third, we can represent all such selections using 5 letters and 2 separators, one to separate the a's from the b's and one to separate the b's from the c's.

For example, aa|bb|c represents the selection of 2 a's, 2 b's and 1 c, whereas |bbb|cc represents 0 a's, 3 b's and 2 c's, and ||ccccc represents 0 a's, 0 b's & 5 c's.

Thus, every such selection consists of 7 symbols, 5 letters & 2 separators. Moreover, every such selection is uniquely determined once we decide upon the placing of the separators within the 7 available positions:

eg. Placing the separators in positions 3 and 6,



determines the selection aa|bb|c

Thus, we have a total of

$$\binom{7}{2} = \frac{7!}{5!\,2!} = 21$$

such selections.

In general, given n distinct objects with repeated selection allowed, we may choose k objects, without regard to order, by deciding upon the placement of $n-1$ separators in $k+n-1$ positions.

Thus, we have a total of $\binom{k+n-1}{n-1}$ such selections.

Eg. A Yes/No vote is taken among 4 people. Considering the outcome as a selection of $k=4$ objects from $n=2$ objects (Yes and No) with repeated selection allowed, we have $\binom{k+n-1}{n-1} = \binom{4+2-1}{2-1} = \binom{5}{1} = 5$ possible outcomes.

In this example, it is feasible to list these outcomes:
we can have 4 No's, 3 No's & 1 Yes, 2 No's & 2 Yes', 1 No & 3 Yes' and 4 Yes'.

Eg. A poll is taken among 10 people, who may respond with either Yes, No or Don't know. Viewing each outcome as a selection of $k=10$ objects from $n=3$ distinct objects with repeated selection gives

$$\binom{k+n-1}{n-1} = \binom{10+3-1}{3-1} = \binom{12}{2} = 66$$

possible poll outcomes.

9

Summarising our discussion of selection k objects from n, we have:

	order imp$^{\underline{t}}$	order not imp$^{\underline{t}}$
repeated sel$^{\underline{n}}$ allowed	n^k	$\binom{k+n-1}{n-1} = \frac{(k+n-1)!}{k!(n-1)!}$
repeated sel$^{\underline{n}}$ not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$