<u>Recall</u>: We have seen that the number of ways of selecting k objects from n is given by:

	<u>order imp^{t}</u>	<u>order not imp^{t}</u>
repeated sel$^{\underline{n}}$ allowed	n^k	$\binom{n-1+k}{n-1}$
repeated sel$^{\underline{n}}$ not allowed	$\frac{n!}{(n-k)!}$	$\frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

We close our discussion of counting by considering one further problem.

We know that there are $n!$ orderings of n distinct objects.
Eg. We can form $5! = 120$ strings of 5 distinct letters from a, b, c, d, e.

It remains to determine the number of orderings of n objects if some repetition is allowed.

Eg How many strings of 7 letters can be formed from the letters in the word BALLOON?

(2)

Examples of such strings include BALLOON, ALLOONB, OLOLNAB, etc.

We can tackle this question by using our knowledge of equivalence relations.

Let's apply subscripts to the letters, whenever necessary, to make them distinct, whereby BALLOON becomes

$$BAL_1L_2O_1O_2N$$

Now, as above, these 7 distinct letters can be ordered in 7! ways.

This number overcounts the true number of orderings as it treats $BAL_1L_2O_1O_2N$ as being different from $BAL_2L_1O_1O_2N$.

We account for this overcounting by using an equivalence relation: two orderings are equivalent if they are the same upon the removal of the subscripts,

eg. $BAL_1L_2O_1O_2N$ is equivalent to $BAL_2L_1O_1O_2N$, $BAL_2L_1O_2O_1N$ and $BAL_1L_2O_2O_1N$.

Indeed, as there are 2! ways to order the L's and 2! ways to order the O's, each equivalence

③

has 4 elements as above.
As these equivalence classes partition the 7! orderings of $BAL_1L_2O_1O_2N$, we have a total of
$\frac{7!}{4}$ different equivalence classes.
Thus, there are $\frac{7!}{4} = 1260$ different orderings of the letters in BALLOON.

Eg. Arguing as above, we have
$\frac{4!}{2} = 12$ different orderings of the letters in AHAB.
We can confirm this by listing them:
AHAB, ABAH, HABA, BAHA,
HBAA, BHAA, BAAH, HAAB,
AHBA, ABHA, AABH, AAHB

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Probability

Probability theory has its origins in the study of games of chance. It began to be formalised and made rigorous in the 17th century by mathematicians such as Fermat, Laplace, etc.

It has applications in a wide range of fields where random processes naturally arise, including artificial intelligence, actuarial and financial mathematics, meteorology, genetics, quantum mechanics, etc.

Some basic definitions:

We'll use the "experiment" to broadly refer to any situation in which outcomes occur randomly. Thus, an experiment might involve flipping coins, rolling dice, making a random selection, etc.

The sample space of an experiment is the universal set of all possible outcomes, and is denoted by Ω (capital omega).

(5)

Thus $\Omega = \{ \omega_i \mid \omega_i$ is an outcome of our experiment $\}$,

outcomes (little omega)

with Ω playing the role of our universal rectangle in a Venn diagram.

We will restrict ourselves to experiments where Ω is a finite set,

i.e. $|\Omega| = n$ for some $n \in \mathbb{N}$,

whereby $\Omega = \{ \omega_1, \omega_2, \ldots, \omega_n \mid \omega_i$ is an outcome of the experiment $\}$

For example, if our experiment consists of flipping a coin, our sample space is

$$\Omega = \{ H , T \}$$

where H denotes a head outcome and T denotes a tail outcome.

If we flip three coins, our sample space is

$$\Omega = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}$$

Eg. If we roll a die our sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ reflecting which side lands face up.

6

If the experiment involves rolling two dice and calculating their sum, our sample space is

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

We are interested in determining the probabilities of events E, which are defined as subsets of Ω.

Thus, an event E is a set of outcomes. Thus, we may be interested in determining the probability of a particular outcome, in the case where $|E| = 1$, but more generally we'll seek to the probability of E where

$$|E| \leq n = |\Omega|.$$

Eg. If our experiment involves rolling and summing two dice, as above, the event that the sum is odd is

$$E = \{3, 5, 7, 9, 11\}.$$

We might be interested in the interaction between events in Ω.

If our exp^{t} involves flipping three coins in order, the event that exactly 2 heads occurs is

$$E_1 = \{HHT, HTH, THH\}.$$

The event that tails comes up for the first time on the second flip is

$$E_2 = \{HTH, HTT\}$$

The event that exactly 2 heads occur <u>or</u> that tails comes up for the first time on the second flip is the <u>union</u>

$$E_1 \cup E_2 = \{HHT, HTH, THH, HTT\}$$

Similarly, the event that exactly 2 heads <u>and</u> that tails comes up for the first time on the second flip is the <u>intersection</u>

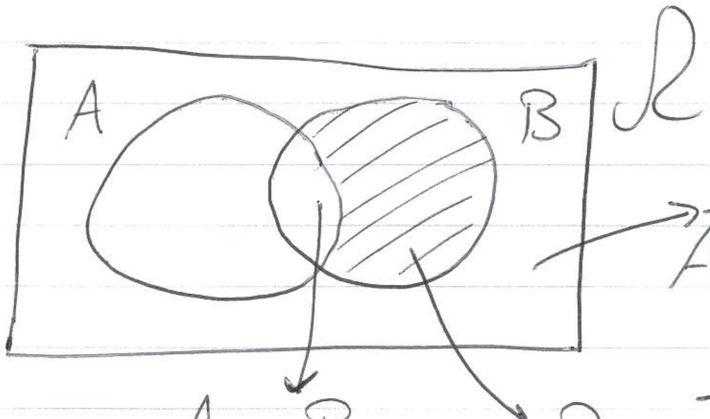
$$E_1 \cap E_2 = \{HTH\}.$$

Generalising the above, we note that we may apply our knowledge of sets and their operations to the study of events.
To aid this process, we have the following dictionary of notation and terminology:

Notation	set language	events language
$A \subseteq \Omega$	A is a subset of Ω	A is an event
$A \cup B$	the union of A & B	A or B
$A \cap B$	the intersection of A & B	A and B
$A \cap B = \emptyset$	A and B are disjoint	A and B are mutually exclusive
$A \subseteq B$	A is a subset of B	A implies B
$\bar{A}$	the complement of A	the complementary event of A

Thus, we can apply Venn diagrams and set identities to our study of events

eg.



$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$A \cap B$ $\qquad$ $B \cap \bar{A}$

As we'll see, we can also apply our knowledge of counting and combinatorics in this regard.

For starters, we recall that there are 2^n events in a sample space Ω with $|\Omega| = n$, including $\emptyset$, the empty set of outcomes, and Ω itself, the set of all possible outcomes.

- A probability measure is a function P which associates to every event E a number $P(E)$ such that $0 \leq P(E) \leq 1$

We use a probability measure to assign probabilities to events. This is done as follows:

To each outcome $\omega_i \in \Omega = \{\omega_1, \ldots, \omega_n\}$, we associate its probability $p(\omega_i)$ in such a way that this number $p(\omega_i) \geq 0$ gives rise to

$$p(\omega_1) + p(\omega_2) + \ldots + p(\omega_n) = 1.$$

Then we define the probability of an event E to be the sum of the probabilities of the outcomes in E,

i.e. for $E = \{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_K}\}$ (a general subset of Ω)

we have $p(E) = p(\omega_{j_1}) + p(\omega_{j_2}) + \ldots + p(\omega_{j_K})$

We use the convention that $p(\phi)=0$.

As a result of the condition that $p(\omega_1)+\dots+p(\omega_n)=1$, it follows that

- $0 \le p(\omega_i) \le 1$ for any outcome ω_i

and

- $p(\Omega)=p(\omega_1)+\dots+p(\omega_n)=1$.

<u>Note</u>: Provided our values $p(\omega_i)$ satisfy $p(\omega_i)\geq 0$ for all i and $p(\omega_1)+\dots+p(\omega_n)=1$, we are free to choose whatever values we like. These choices might be random, or based on subjective feelings or empirical evidence.

In our classical examples of experiments, its clear as to what the probabilities $p(\omega_i)$ should be.

<u>Eg.</u> When flipping a coin, it's reasonable to assume that we're dealing with a fair coin unless told otherwise, whereby

$$p(H)=\tfrac{1}{2} \text{ and } p(T)=\tfrac{1}{2}.$$

A biased coin that falls heads twice as often as tails gives

$$p(H)=\tfrac{2}{3} \text{ and } p(T)=\tfrac{1}{3}.$$