MS115 Mathematics for Enterprise Computing

Tutorial Sheet 2 Solutions

1.

P	Q	not P	not Q	$P \vee Q$	not $(P \vee Q)$	not $P \wedge$ not Q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

2. (i) Here P is the statement that n and m are even integers and Q is the statement that $n+m$ is an even integer. We will assume P and deduce Q.

   We have that $n = 2k$ and $m = 2q$ for some integers k and q. Hence $n+m = 2k+2q = 2(k+q)$, establishing that $n+m$ is an even integer.

   (ii) Here P is the statement that n^2 is an even integer and Q is the statement that n is an even integer. In the contrapositive argument, we assume not Q and seek to deduce not P. Thus, we assume that n is not an even integer and seek to deduce that n^2 is not an even integer.

   We therefore have that n is an odd integer, with $n = 2k+1$ for some integer k. Hence,

$$n^2 = (2k+1)(2k+1) = 4k^2 + 2k + 2k + 1 = 2(2k^2 + 2k) + 1,$$

   whereby n^2 is an odd integer.

3. The "base case" of our induction is where $n = 1$ (as we are asked to prove the statement for all $n \geq 1$). Here, as $2(1) - 1 = 1$ and $1^2 = 1$, $P(1)$ states that $1 = 1$, which is true.

   We will show that $P(n) \Rightarrow P(n+1)$ using a direct argument.

   We assume that $P(n)$ is true, whereby $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

   The L.H.S. of $P(n+1)$ is $1 + 3 + 5 + \ldots + (2n-1) + (2(n+1) - 1)$.

   As $P(n)$ is true, the L.H.S. of $P(n+1)$ is $n^2 + (2(n+1) - 1)$. Now

$$n^2 + (2(n+1) - 1) = n^2 + (2n + 2 - 1) = n^2 + 2n - 1 = (n+1)^2,$$

   which is the RHS of $P(n+1)$.

4. (*i*) The "base case" of our induction is where $n = 1$ (as we are asked to prove the statement for all $n \geq 1$). Here, as $a^{1-1} = a^0 = 1$ and $\frac{1-a^1}{1-a} = \frac{1-a}{1-a} = 1$, $P(1)$ states that $1 = 1$, which is true.

We will show that $P(n) \Rightarrow P(n+1)$ using a direct argument.

We assume that $P(n)$ is true, whereby

$$1 + a + a^2 + \ldots + a^{n-1} = \frac{1-a^n}{1-a}.$$

The L.H.S. of $P(n+1)$ is

$$1 + a + a^2 + \ldots + a^{n-1} + a^{(n+1)-1} = 1 + a + a^2 + \ldots + a^{n-1} + a^n.$$

As $P(n)$ is true, the L.H.S. of $P(n+1)$ is $\frac{1-a^n}{1-a} + a^n$. Now

$$\frac{1-a^n}{1-a} + a^n = \frac{1-a^n + a^n(1-a)}{1-a} = \frac{1-a^n+a^n-a^{n+1}}{1-a} = \frac{1-a^{n+1}}{1-a},$$

which is the RHS of $P(n+1)$.

(*ii*) The "base case" of our induction is where $n = 1$ (as we are asked to prove the statement for all $n \geq 1$). Here, as $1^3 - 1 = 0$, $P(1)$ states that 0 is divisible by 3, which is true as $0 = 3k$ for $k = 0$.

We will show that $P(n) \Rightarrow P(n+1)$ using a direct argument.

We assume that $P(n)$ is true, whereby $n^3 - n$ is divisible by 3. Thus $n^3 - n = 3k$ for some integer k.

$P(n+1)$ states that $(n+1)^3 - (n+1)$ is divisible by 3. Thus we seek to deduce that $(n+1)^3 - (n+1) = 3m$ for some integer m.

Now, multiplying out the left hand side of this equation, we have that

$$(n+1)^3 - (n+1) = (n^3 + 3n^2 + 3n + 1) - (n+1) = n^3 + 3n^2 + 2n.$$

As usual, we seek to relate the $P(n+1)$ to $P(n)$. We can do this by adding and subtracting n:

$$n^3 + 3n^2 + 2n = (n^3 - n) + (3n^2 + 2n + n).$$

As $P(n)$ is true, we know that $n^3 - n = 3k$ for some integer k.

The L.H.S. of $P(n+1)$ now becomes

$$(n^3 - n) + (3n^2 + 2n + n) = (3k) + (3n^2 + 3n) = 3(k + n^2 + n).$$

Thus, we have dedcued $P(n+1)$.