MS115 Mathematics for Enterprise Computing

Tutorial Sheet 4

1. For each of the following relations on $\mathbb{Z}$, determine whether the relation is (a) reflexive, (b) symmetric, (c) transitive.

   (i) xRy exactly when $x+y$ is an odd integer.

   (a) Not reflexive: We do not have that x is related to x for all $x \in \mathbb{Z}$. Indeed, x is not related to x for any x as $x+x=2x$, which is even.

   (b) Symmetric: If $x+y=2k+1$ for some $k \in \mathbb{Z}$, then $y+x=2k+1$.

   (c) Not transitive: eg. $1R2$ and $2R1$ but 1 is not related to 1.

   (ii) xRy exactly when $x+y$ is an even integer.

   (a) Reflexive: x is related to x for all $x \in \mathbb{Z}$ as $x+x=2x$, which is even.

   (b) Symmetric: If $x+y=2k$ for some $k \in \mathbb{Z}$, then $y+x=2k$.

   (c) Transitive: If $x+y=2k$ and $y+z=2\ell$ for some $k, \ell \in \mathbb{Z}$ then

   $$(x+y)+(y+z)=2k+2\ell \quad \text{whereby} \quad x+z=2(k+\ell-y).$$

2. Let n be a fixed positive integer. Consider the relation R on $\mathbb{Z}$ defined by xRy exactly when $x-y$ is divisible by n.

   (i) Prove that R is an equivalence relation on $\mathbb{Z}$.

   (a) Reflexive: x is related to x for all $x \in \mathbb{Z}$ as $x-x=0=0(n)$.

   (b) Symmetric: If $x-y=nk$ for some $k \in \mathbb{Z}$, then $y-x=n(-k)$.

   (c) Transitive: If $x-y=nk$ and $y-z=n\ell$ for some $k, \ell \in \mathbb{Z}$ then $(x-y)+(y-z)=nk-n\ell$, whereby $x-z=n(k-\ell)$.

   (ii) Express the relation xRy in terms of x and y sharing a common property.

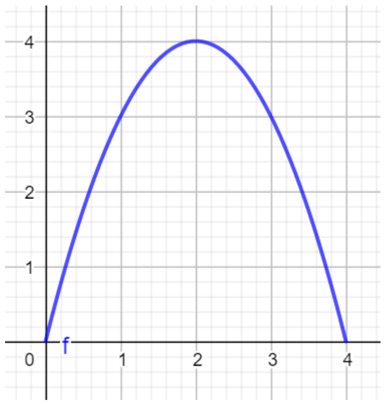
   Here, x is related to y exactly when x and y have the same remainder after division by n.

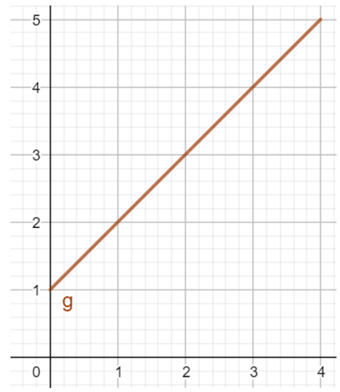
   (iii) Determine the number of equivalence classes in the associated partition of $\mathbb{Z}$.

   There are n equivalence classes: $E_0, E_1, \ldots, E_{n-1}$.

3. The *graph* of a function f is a graphical representation of all ordered pairs $(x, f(x))$ for x an element of the domain of f.

   Consider the following graphs of two functions f and g:





   (i) Determine the domain and range of f.

   Referencing the horizontal axis, the domain of f is the interval $[0, 4]$.

   Referencing the vertical axis, the range of f is the interval $[0, 4]$.

   (ii) Determine the domain and range of g.

   Referencing the horizontal axis, the domain of f is the interval $[0, 4]$.

   Referencing the vertical axis, the range of f is the interval $[1, 5]$.

   (iii) Justifying your answer, determine whether f is invertible.

   Here, it is understood that f is a function mapping $[0, 4]$ to $[0, 4]$.

   f is not invertible. We do have that every $b \in [0, 4]$ is the image of some $a \in [0, 4]$. However, there exist $b \in [0, 4]$ that are not the image of exactly one $a \in [0, 4]$. For example, $f(1) = 3$ and $f(3) = 3$, whereby $f^{-1}(3)$ is not defined.

   (iv) Justifying your answer, determine whether g is invertible.

   Here, it is understood that g is a function mapping $[0, 4]$ to $[1, 5]$.

   g is invertible as every $b \in [1, 5]$ is the image of exactly one $a \in [0, 4]$.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 1$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 3x + 3$.

   (i) Determine the output of the function $g \circ f$.

   $g \circ f(x) = g(f(x)) = g(2x - 1) = 3(2x - 1) + 3 = 6x.$

   (ii) Determine the output of the function $f \circ g$.

   $f \circ g(x) = f(g(x)) = f(3x + 3) = 2(3x + 3) - 1 = 6x + 5.$