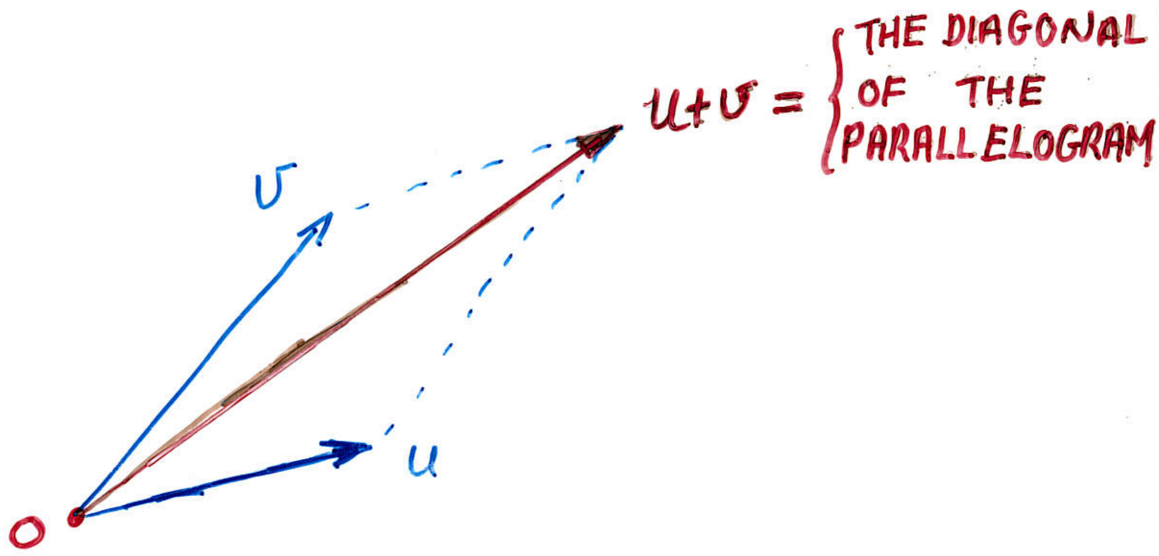
VECTORS IN SPACE

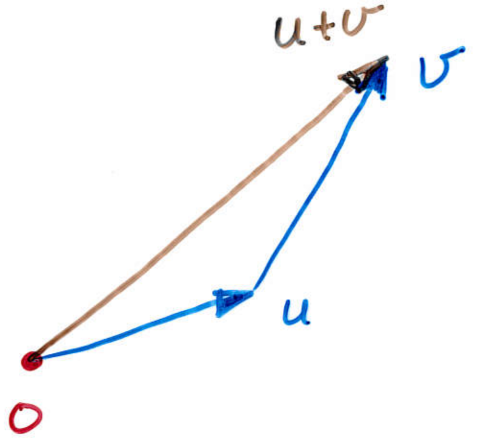
Fix a point "O" in space ("O" will be called the origin). By a VECTOR IN SPACE we mean an "ARROW" in space whose base is at O

ADDITION: We add two vectors u and v by the PARALLELOGRAM LAW:



Clearly $u+v = v+u$ since each sum is the diagonal of the parallelogram shown. Thus vector addition is COMMUTATIVE.

REMARK: For convenience of addition we sometimes move vectors by parallel translation so that the base of one lies on the tip of another:



"TRIANGLE LAW"

This is just to facilitate addition but, be in no doubt, the base of all vectors are at "o"

SCALAR MULTIPLICATION:

For each $\lambda \in \mathbb{R}$ and for each vector v

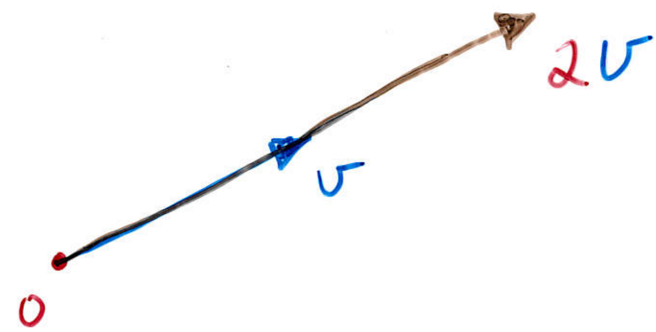
λv is the vector such that

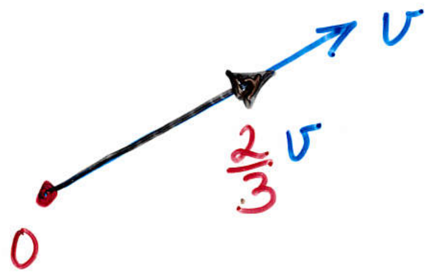
(i) $\text{LENGTH}(\lambda v) = |\lambda| \, \text{LENGTH}(v)$

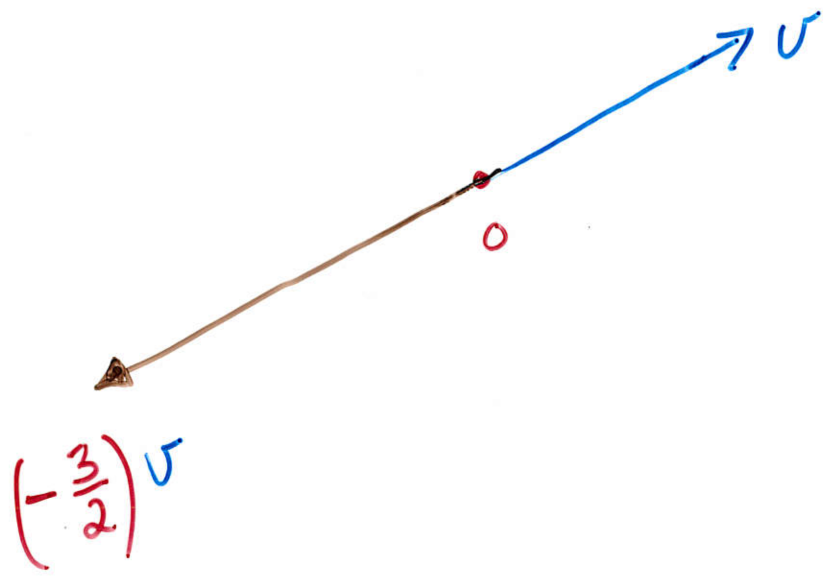
and

(ii) $\text{DIRECTION}(\lambda v) = \begin{cases} \text{DIRECTION}(v) & \text{if } \lambda > 0 \\ \text{OPPOSITE DIRECTION}(v) & \text{if } \lambda < 0 \end{cases}$

NOTE: It is because of property (i) above that $\lambda \in \mathbb{R}$ used in this way (i.e. to multiply a vector) is called a SCALAR.







REMARKS:

(i) The ZERO VECTOR is (by definition) the ORIGIN. Thus by property (i) of scalar multiplication

$$0 v = 0$$

$0 \in \mathbb{R}$

The ZERO VECTOR

(ii) By property (ii) of scalar multiplication together with the parallelogram law for addition we have

$$(-1) v + v = 0$$

so that

$$(-1) v = -v .$$

(iii) It should be clear also from the definitions above that

$$\lambda(u + v) = \lambda u + \lambda v$$

To see this, use what you know about SIMILAR TRIANGLES

and

$$(\alpha + \beta) u = \alpha u + \alpha u$$

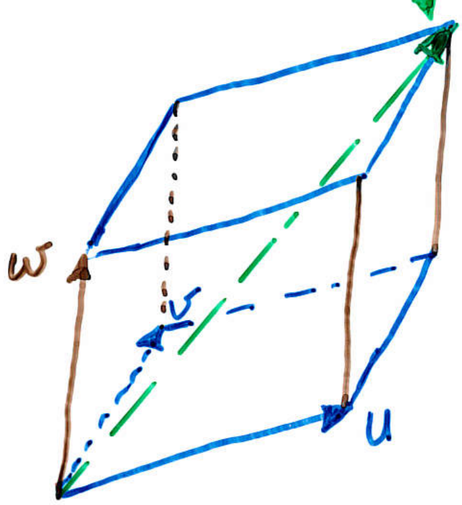
ASSOCIATIVITY OF ADDITION :

Given any three vectors u, v and w, we can add them in two different ways to get

$$(u+v)+w$$

or

$$u+(v+w)$$



$(u+v)+w = u+(v+w)$ is the DIAGONAL of the parallel pippet formed by u, v, w

Thus for all vectors u, v, w we have

$$(u+v)+w = u+(v+w)$$

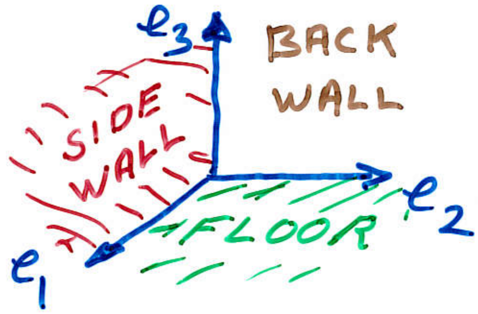
so that VECTOR ADDITION is ASSOCIATIVE and in particular we can write

$u+v+w$ without any ambiguity in its meaning.

FROM ARROWS TO ALGEBRA

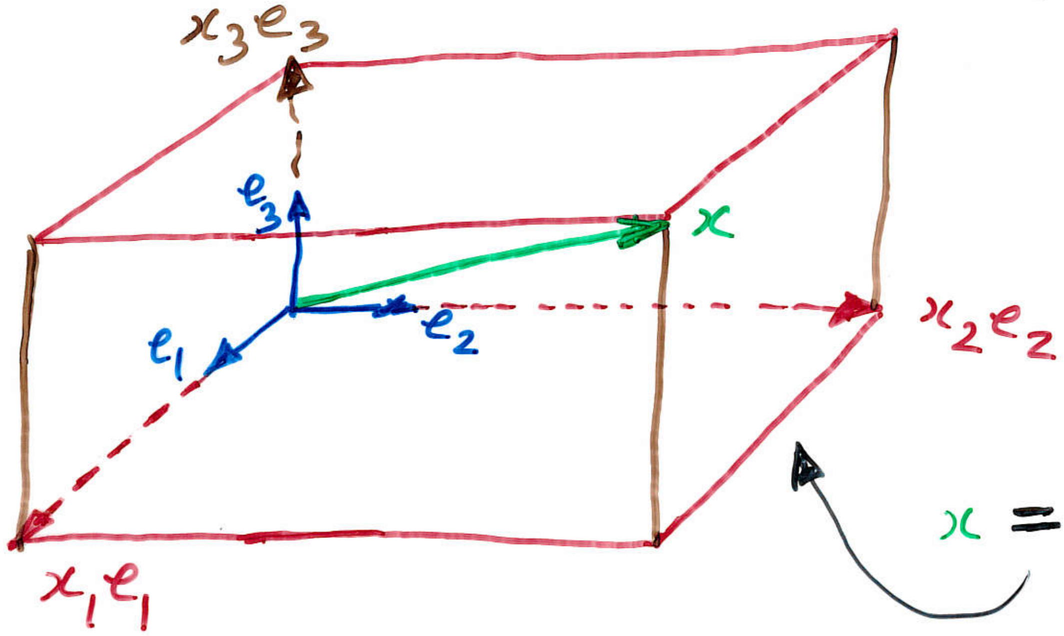
To make the operations of vector addition and scalar multiplications managable we proceed as follows:

Fix THREE MUTUALLY PERPENDICULAR VECTORS e_1, e_2, e_3 such that EACH HAS UNIT LENGTH



Vectors such as e_1, e_2, e_3 are said to form AN ORTHONORMAL FRAME

Relative to $e_1, e_2\ e_3$ any vector x can be written UNIQUELY in the form:



$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

REMARK: In writing

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

we are already making use of the associativity of vector addition.

In any case, to spare on ink we will write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{to mean} \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

Now addition takes the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = (x_1 e_1 + x_2 e_2 + x_3 e_3) + (y_1 e_1 + y_2 e_2 + y_3 e_3)$$

Associative law

$$= x_1 e_1 + x_2 e_2 + x_3 e_3 + y_1 e_1 + y_2 e_2 + y_3 e_3$$

After much use of the commutative and associative laws

$$= x_1 e_1 + y_1 e_1 + x_2 e_2 + x_3 e_3 + y_2 e_2 + y_3 e_3$$

$$= (x_1 + y_1) e_1 + x_2 e_2 + x_3 e_3 + y_2 e_2 + y_3 e_3$$

$$\vdots$$

$$= (x_1 + y_1)e_1 + (x_2 + y_2)e_2 + (x_3 + y_3)e_3$$

$$= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

Similarly

$$\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$

As a consequence of these two formula we have in general that

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}$$

<u>EXAMPLE:</u>

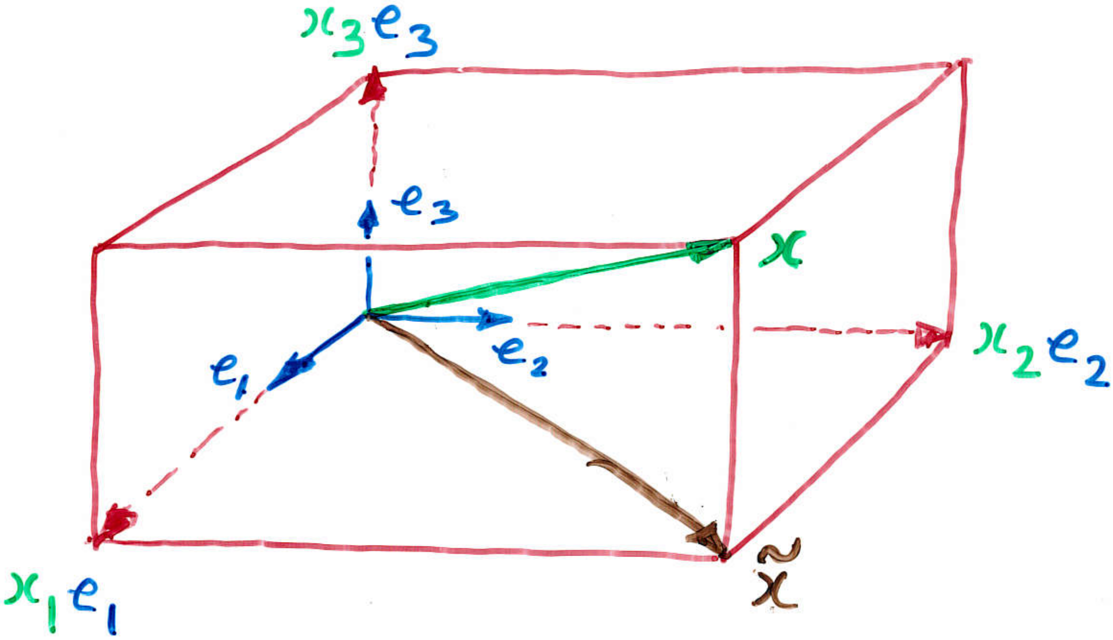
$$3 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 + 2 \\ -3 - 4 \\ 12 - 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 8 \end{bmatrix}$$

THE CALCULATION OF LENGTHS

NOTATION: From now on, for any vector x we will write

$\|x\|$ to denote the LENGTH OF x

Now, for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$



we have from the Theorem of Pythagoras that

$$\|x\|^2 = \|\tilde{x}\|^2 + \|x_3 e_3\|^2$$

$$= \|\tilde{x}\|^2 + \left(|x_3|\,\|e_3\|\right)^2$$

$$= \|\tilde{x}\|^2 + x_3^2 \qquad \left\{ \begin{array}{l} \text{since} \\ \|e_3\| = 1 \end{array} \right.$$

$$= \|x_1 e_1 + x_2 e_2\|^2 + x_3^2$$

Again by Pythagoras since $e_1 \perp e_2$

$$= \|x_1 e_1\|^2 + \|x_2 e_2\|^2 + x_3^2$$

$$= x_1^2 + x_2^2 + x_3^2$$

since $\|e_1\| = \|e_2\| = 1$

So finally we have that

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \Longrightarrow \boxed{\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}}$$

EXAMPLE: For $x = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$$\|x\| = \sqrt{(-2)^2 + 1^2 + 3^2}$$

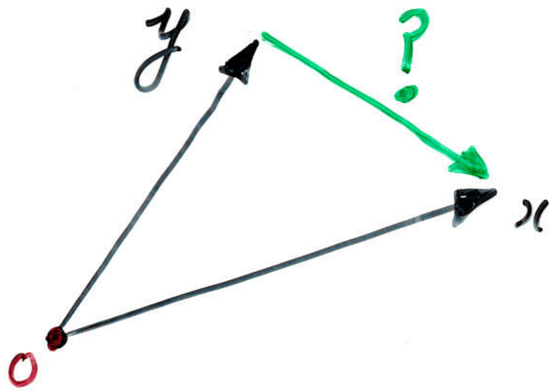
$$= \sqrt{4 + 1 + 9}$$

$$= \sqrt{14}$$

THE DISTANCE BETWEEN VECTORS

For any vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

observe the picture



By the triangle law for addition

$$y + ? = x$$

$$\Rightarrow \quad ? = x - y$$

and it should be obvious that $\|?\|$ is the distance between x and y.

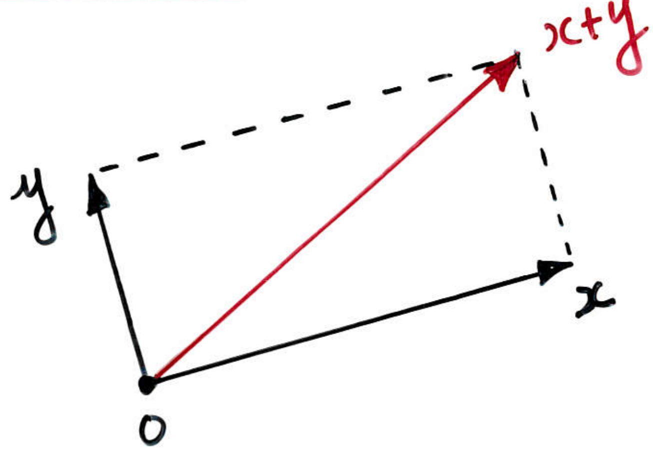
Thus

$$\boxed{\text{DISTANCE FROM } x \text{ TO } y} = \|x - y\|$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

CALCULATION OF ANGLES

To begin with, we recall what is known to all from secondary school, namely:



$\boxed{\text{vectors } x \,\&\, y \text{ are PERPENDICULAR}} \iff \boxed{\text{The Theorem of Pythagoras holds}}$

Here we mean $x \neq 0$ & $y \neq 0$

$$\iff \boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2}$$

$$\iff \boxed{(x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 = \|x\|^2 + \|y\|^2}$$

$$\iff \boxed{\|x\|^2 + 2(x_1y_1 + x_2y_2 + x_3y_3) + \|y\|^2 = \|x\|^2 + \|y\|^2}$$

$$\iff \boxed{x_1y_1 + x_2y_2 + x_3y_3 = 0}$$

EXAMPLE: Are the vectors

$$x = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$$ perpendicular

to one another?

SOLUTION: Here

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = (-2)(5) + (3)(4) + (1)(-3)$$
$$= -1 \neq 0$$

so that x & y are NOT mutually perpendicular

DEFINITION For any vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(which may be zero) we DEFINE

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

and call it THE INNER PRODUCT OF x AND y.

PROPERTIES OF THE INNER PRODUCT

(i) $\langle x, y \rangle \in \mathbb{R}$

(ii) $\langle x, y \rangle = \langle y, x \rangle$

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

(iv) $\langle x, x \rangle = \|x\|^2 \geq 0$

with equality $\Leftrightarrow x = 0$

NOTE: All of these properties are completely obvious. However, in case this is not so for some, let's

PROVE PART (iii):

$$\langle \alpha x + \beta y, z \rangle = \left\langle \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right\rangle$$

$$= (\alpha x_1 + \beta y_1) z_1 + (\cdots\cdots) z_2 + (\cdots) z_3$$

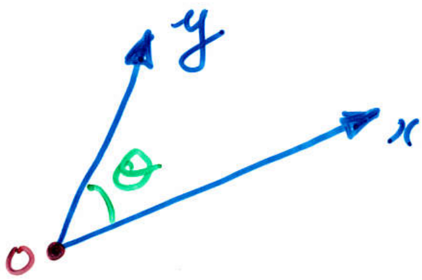
$$= \alpha (x_1 z_1 + x_2 z_2 + x_3 z_3) + \beta (y_1 z_1 + \ldots)$$

$$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$

THEOREM: For any NON-ZERO vectors x and y we have that

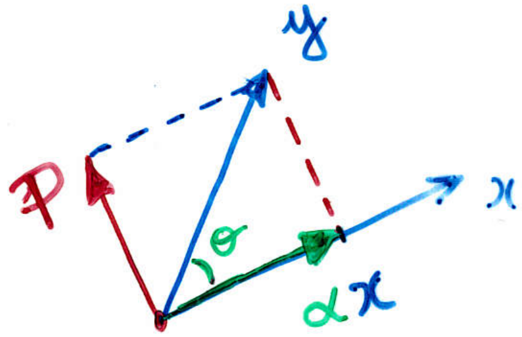
$$\|x\| \, \|y\| \cos\theta = \langle x, y \rangle$$

where θ is the ANGLE BETWEEN x and y.



Proof: Write the vector y in the form

$$\boxed{y = \alpha x + P}$$



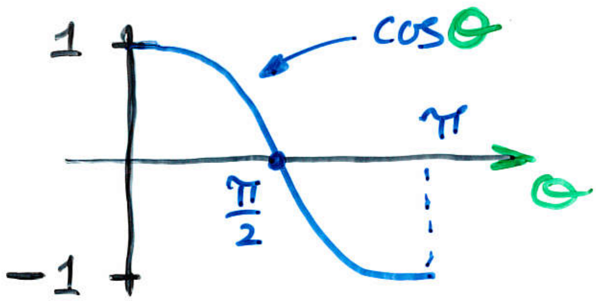
where $\alpha \in \mathbb{R}$ and P is perpendicular to x.

NOTE: The angle $\theta \in [0, \pi]$ and

$$\boxed{\theta \in \left[0, \tfrac{\pi}{2}\right]} \iff \boxed{\alpha \geq 0}$$

$$\Updownarrow$$

$$\boxed{\cos\theta \geq 0}$$



Now,

$$y = \alpha x + P$$

$$\Rightarrow \langle x, y\rangle = \langle x, \alpha x + P\rangle$$

$$= \alpha \underbrace{\langle x, x\rangle}_{\|x\|^2} + \underbrace{\langle x, P\rangle}_{0}$$

$$= \alpha \|x\|^2$$

$$\Rightarrow \boxed{\alpha = \frac{\langle x, y\rangle}{\|x\|^2}} \quad \cdots\cdots\cdots (1)$$

On the other hand (from the diagram)

$$\cos\theta = \frac{(\operatorname{sign}\alpha)\,\|\alpha x\|}{\|y\|}$$

$$= \frac{\alpha\|x\|}{\|y\|}$$

By (1) above

$$= \frac{\langle x, y\rangle}{\|x\|^2}\cdot\frac{\|x\|}{\|y\|} = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}.$$

TERMINOLOGY: The quantity

$$\|x\| \, \|y\| \cos\theta$$

is called THE DOT PRODUCT of the vectors x and y. However, its calculation is done ONLY by means of the formula

$$\|x\| \, \|y\| \cos\theta = \langle x, y \rangle$$

(which we have just established). Thus it is the INNER PRODUCT WHICH IS CALCULATED.

APPLICATIONS:

[1] Find the ANGLE between the vectors

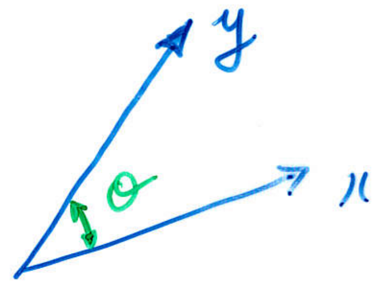
$$x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

SOLUTION:

$$\|x\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

$$\|y\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

$$\langle x, y \rangle = (1)(2) + (2)(1) + (-1)(1) = 3.$$



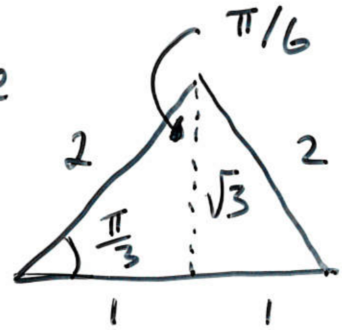
Now,

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|} = \frac{3}{6} = \frac{1}{2}.$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Aside



$$\cos\frac{\pi}{3} = \frac{1}{2}.$$

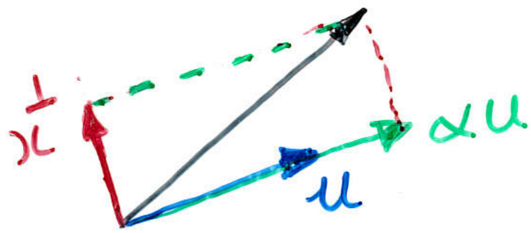
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

[2] In the case of the vectors

$$x = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad u = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

find the constant $\alpha \in \mathbb{R}$ and the vector $x^{\perp}$ which is PERPENDICULAR TO THE UNIT VECTOR u such that

$$x = \alpha u + x^{\perp}$$



SOLUTION:

$$x = \alpha u + x^{\perp}$$

$$\Rightarrow \langle x, u \rangle = \alpha \underbrace{\langle u, u \rangle}_{\substack{\| \\ 1}} + \underbrace{\langle x^{\perp}, u \rangle}_{\substack{\| \\ 0}}$$

since $\|u\| = 1$

since $x^{\perp} \perp u$

$$\Rightarrow \alpha = \langle x, u \rangle$$

In this example

$$\alpha = \left\langle \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{3}}(-3-1+2) = -\frac{2}{\sqrt{3}}.$$

To find $x^{\perp}$, observe that

$$x = \alpha u + x^{\perp}$$

$$\Rightarrow \boxed{x^{\perp} = x - \alpha u}$$

So in this example:

$$x^{\perp} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} - \left(-\frac{2}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

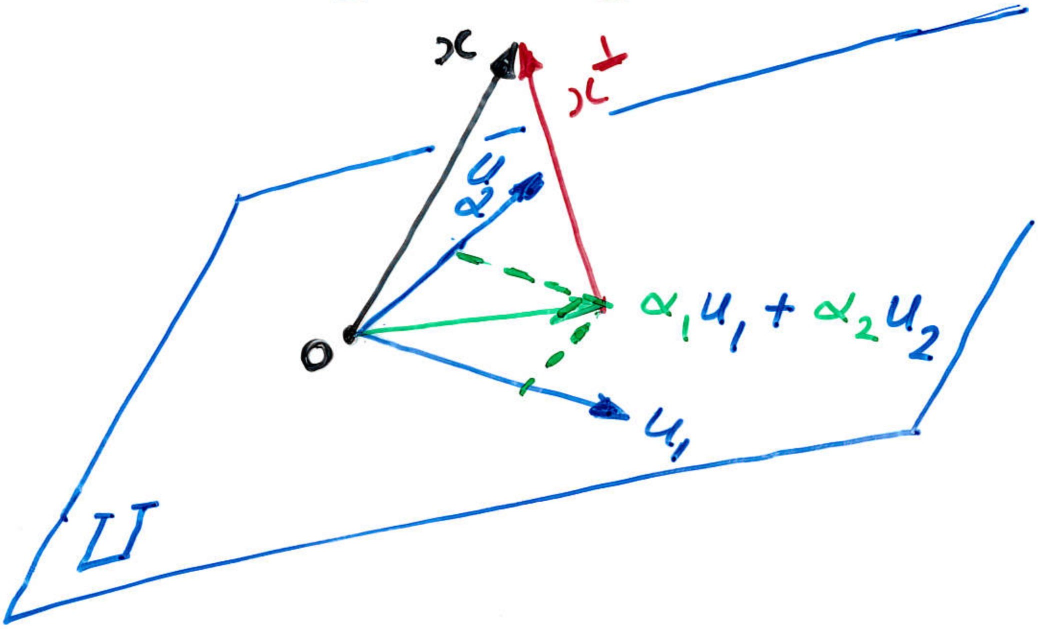
$$= \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Be CAREFUL with the arithmetic at this step

$$= \frac{1}{3}\left\{\begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}\right\}$$

$$= \frac{1}{3}\begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix}.$$

[3] Let the vectors u_1, u_2 be mutually perpendicular and let each be of unit length. Denote by Π the PLANE THROUGH THE ORIGIN WHICH CONTAINS u_1 and u_2



For any FIXED vector "x" find the constants $\alpha_1, \alpha_2 \in \mathbb{R}$ and the vector $x^\perp$ WHICH IS PERPENDICULAR TO THE PLANE Π such that

$$x = \underbrace{\alpha_1 u_1 + \alpha_2 u_2}_{\text{THIS VECTOR IS ON THE PLANE } \Pi} + x^\perp$$

SOLUTION:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + x^{\perp}$$

$$\Rightarrow \langle x, u_1 \rangle = \alpha_1 \underbrace{\langle u_1, u_1 \rangle}_{=1} + \alpha_2 \underbrace{\langle u_2, u_1 \rangle}_{=0} + \underbrace{\langle x^{\perp}, u_1 \rangle}_{=0}$$

since u_1 has unit length

since $u_1 \perp u_2$

since $x^{\perp} \perp u_1$

$$\Rightarrow \boxed{\alpha_1 = \langle x, u_1 \rangle}$$

and similarely

$$\boxed{\alpha_2 = \langle x, u_2 \rangle}$$

Thus

$$x = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + x^{\perp}$$

and

$$\boxed{x^{\perp} = x - \langle x, u_1 \rangle u_1 - \langle x, u_2 \rangle u_2}$$

TERMINOLOGY: In the context above, the vector

$$\boxed{\langle x, u_1\rangle u_1 + \langle x, u_2\rangle u_2}$$

is called THE ORTHOGONAL PROJECTION OF x ONTO THE PLANE Π. Clearly it is the VECTOR IN Π WHICH IS NEAREST TO x (THIS FOLLOWS FROM PYTHAGORAS).

The vector $x^{\perp}$ is called THE COMPONENT OF x WHICH IS PERPENDICULAR TO THE PLANE Π.