Chapter 2: Calculus of Parametrized curves.

GENERALITIES:

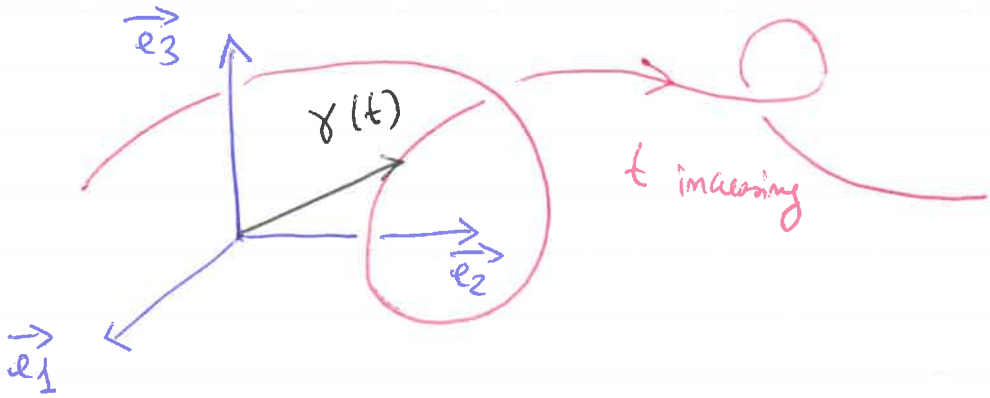
DEFINITION: (Parametrization of a curve)

A locally one-to-one map

$$\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3 : t \mapsto \vec{\gamma}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

is called a parametrization of the curve

$$\mathcal{C} := \left\{ \vec{\gamma}(t) \in \mathbb{R}^3 \text{ s.t. } t \in \mathbb{R} \right\}$$



REMARKS:

- $\vec{\gamma}$ is a mapping while $\mathcal{C}$ is a subset of $\mathbb{R}^3$.
- The variable "t" is called the parameter which we usually think of as "time". Thus we think of $\mathcal{C}$ as a curve in space along

which a FLY is flying so that

$\vec{\gamma}(t)$ = the position of this fly at time t.

- Of course there is very little unique about such a parametrization, it all depends on the "choice of fly".

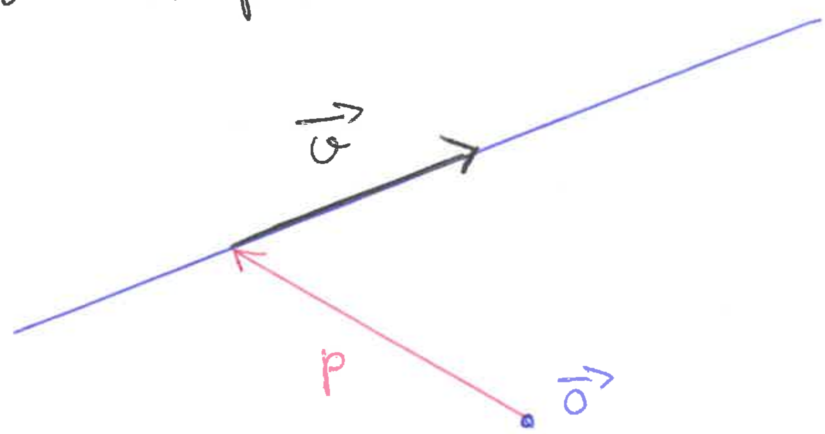
- We will allow ourselves to consider

$$\vec{\gamma} : I \to \mathbb{R}^3 \quad \text{where } I \text{ is an interval } \subset \mathbb{R},$$

when useful.

EXAMPLE: We've already seen parametrizations of lines,

$$\vec{\gamma}(t) = p + t\vec{v}$$



LIMITS:

DEFINITIONS:

Let us consider $f: \mathbb{R} \to \mathbb{R}$
$t \mapsto f(t)$.

We want to formalize the following definition:

" We say that

$$\lim_{t \to t_0} f(t) = \ell \overset{(def)}{\iff} \boxed{\begin{array}{c} t \text{ close enough to } t_0 \\ \Downarrow \\ f(t) \text{ close to } \ell \end{array}} \text{"}$$

The above qualitative statement is made quantitative as follows:

" For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - t_0| < \delta \implies |f(t) - \ell| < \varepsilon \text{"}$$

We extend the definition to $\vec{\gamma}: \mathbb{R} \to \mathbb{R}^3$
$t \mapsto \vec{\gamma}(t)$

as follows:

$\lim_{t \to t_0} \vec{\gamma}(t) = \vec{\ell} \overset{(def)}{\iff}$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|t - t_0| < \delta \implies \|\vec{\gamma}(t) - \vec{\ell}\| < \varepsilon$.

REMARKS:

- ε is to be seen as "as little as wanted" and is some kind of exigency.

  δ is a treshold: if $|t-t_0|<\delta$, then our exigency is satisfied.

- notice that
$$|t-t_0| = d(t,t_0),$$
$$|f(t)-\ell| = d(f(t),\ell),$$
$$\|\vec{\gamma}(t)-\vec{\ell}\| = d(\vec{\gamma}(t),\vec{\ell})$$
are distances

- This definition takes different shapes in other contexts, such as:

$$\lim_{t\to+\infty}\vec{\gamma}(t)=\vec{\ell} \iff \text{For every } \varepsilon>0, \text{ there exists } n>0 \text{ such that } t\geq n \Rightarrow \|\vec{\gamma}(t)-\vec{\ell}\|<\varepsilon.$$

(This definition is like this because $d(t,+\infty)=+\infty$ is difficult to use.)

- We have

$$\lim_{t\to t_0}\vec{\gamma}(t)=\vec{\ell} \iff \begin{bmatrix} \lim_{t\to t_0} x(t) = x_\ell \\ \lim_{t\to t_0} y(t) = y_\ell \\ \lim_{t\to t_0} z(t) = z_\ell \end{bmatrix}$$

EXAMPLE:

$$\lim_{t \to -2} \begin{bmatrix} t+4 \\ t^2-1 \\ t^3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}.$$

CONTINUITY:

We say that $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(t)$

is CONTINOUS AT t_0 $\overset{(\text{def})}{\iff}$ $\lim_{t \to t_0} f(t) = f(t_0)$.

- That is to say f is "where we expect" when $t = t_0$.
- The definition generalizes to $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$.
- Another interpretation of continuity is: the graph of f (the curve $\mathcal{C}$ associated to $\vec{\gamma}$) can be drawn without "lifting the pen from the paper".

DIFFERENTIATION :

DEFINITION : For $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$, $t \mapsto \vec{\gamma}(t)$, we define

the derivative of $\vec{\gamma}$ at $t = t_0$ as follows :

$$\left.\frac{d\vec{\gamma}}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{\vec{\gamma}(t_0 + \Delta t) - \vec{\gamma}(t_0)}{\Delta t}$$

INTERPRETATION :

- $\frac{d\vec{\gamma}}{dt}$ is the speed vector at $\vec{\gamma}(t_0)$: notice that

  $\vec{\gamma}(t_0 + \Delta t) - \vec{\gamma}(t_0)$ is the variation of position between the times t and $t + \Delta t$.

  Δt is the time elapsed between t_0 and $t_0 + \Delta t$.

- $\frac{d\vec{\gamma}}{dt}$ is tangent to the curve $\mathcal{C}$ (associated to $\vec{\gamma}$) at $\vec{\gamma}(t_0)$.

ACCELERATION VECTOR:

Notice that

$$\frac{d\vec{\gamma}}{dt} : \mathbb{R} \longrightarrow \mathbb{R}^3 \qquad \text{is a parametrized curve.}$$
$$t_0 \longmapsto \left.\frac{d\vec{\gamma}}{dt}\right|_{t_0}$$

We can now consider

$$\left.\frac{d}{dt}\left(\frac{d\vec{\gamma}}{dt}\right)\right|_{t_0}, \text{ which we denote by}$$
$$\left.\frac{d^2\vec{\gamma}}{dt^2}\right|_{t_0}.$$

It is the second derivative of $\vec{\gamma}$, and is to be interpreted as the acceleration vector of a particule following $\vec{\gamma}(t)$.

REMARKS: • The quantity $\dfrac{\vec{\gamma}(t_0+\Delta t)-\vec{\gamma}(t_0)}{\Delta t}$ may not have a limit when $\Delta t \to 0$.

If it does, we say that $\vec{\gamma}$ is differentiable at $t = t_0$.

In this class, we will consider curves, functions, ...

, objects such as the derivatives we consider exist. HOWEVER, in the mathematical world, and even in the physical world, it may happen that derivatives exist or not.

- These definitions are valid for $f: \mathbb{R} \to \mathbb{R}$.

- Notations:

We may write $\left.\frac{d\vec{\gamma}}{dt}\right|_{t_0}$ $\dot{\vec{\gamma}}(t_0)$ (when t is time, specifically).

LEIBNIZ

NEWTON

You will find $\frac{d^2\vec{\gamma}}{dt^2} = \ddot{\vec{\gamma}}(t_0)$ in this context.

For functions $f: \mathbb{R} \to \mathbb{R}$ we often write

$$\left.\frac{df}{dt}\right|_{t_0} = f'(t), \quad \frac{d^2 f}{dt^2} = f''(t) \ \ldots$$

LAGRANGE

COMPUTING DERIVATIVES

Exactly like for limits we have

for $\vec{\gamma}: t \to \mathbb{R}^3$

$$t \mapsto \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix},$$

$$\left.\frac{d\vec{\gamma}}{dt}\right|_{t_0} = \begin{bmatrix} \left.\frac{dx}{dt}\right|_{t_0} \\ \left.\frac{dy}{dt}\right|_{t_0} \\ \left.\frac{dz}{dt}\right|_{t_0} \end{bmatrix},$$

so that we only need to understand how to differentiate functions.

In practice:

→ We know the derivatives of usual functions.

→ There are rules to "cook" with these usual functions.

DERIVATIVES OF USUAL FUNCTIONS:

$f(x)$	$\left.\frac{df}{dx}\right\|_x$
x^n	nx^{n-1}
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1+\tan^2(x)$

REMARK: This is a "first version", we will see later that we can enhance it.

FIRST "COOKING RULES"

DERIVATIVE OF A SUM

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

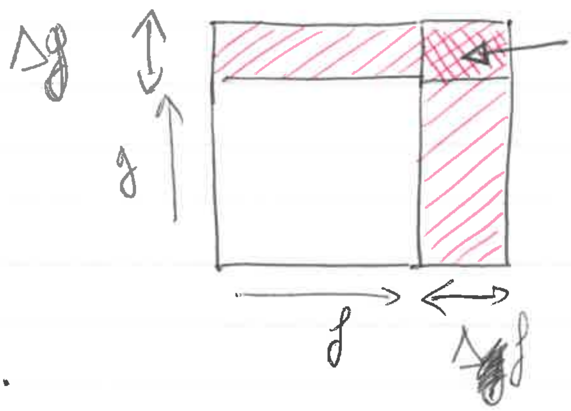
- Also true for vector-valued functions:

$$\frac{d}{dt}\left(\vec{\gamma}_1 + \vec{\gamma}_2\right) = \frac{d\vec{\gamma}_1}{dt} + \frac{d\vec{\gamma}_2}{dt}.$$

DERIVATIVE OF A PRODUCT

$$\frac{d}{dx}(f\cdot g) = \left(\frac{df}{dx}\right)\cdot g + f\cdot\left(\frac{dg}{dx}\right)$$

Intuition: $\Delta(f\cdot g) \simeq (\Delta f)\cdot g + f\cdot(\Delta g)$



when we take the limit $\Delta t \to 0$ this part will be negligible.

▨ : $\Delta(f\cdot g)$.

This formula stays true for the inner and the outer products:

$$\frac{d}{dt}\left(\langle \vec{\gamma_1}(t), \vec{\gamma_2}(t) \rangle \right)$$

$$= \langle \frac{d}{dt}\vec{\gamma_1}(t), \vec{\gamma_2}(t) \rangle + \langle \vec{\gamma_1}(t), \frac{d}{dt}\vec{\gamma_2}(t) \rangle$$

and

$$\frac{d}{dt}\left(\vec{\gamma_1}(t) \times \vec{\gamma_2}(t) \right)$$

$$= \left(\frac{d}{dt}\vec{\gamma_1}(t)\right) \times \vec{\gamma_2}(t) + \vec{\gamma_1}(t) \times \left(\frac{d}{dt}\vec{\gamma_2}(t)\right).$$

IDEE OF PROOF:

Notice that $\langle \vec{\gamma_1}(t), \vec{\gamma_2}(t) \rangle$

$$= x_1(t)x_2(t) + y_1(t) \cdot y_2(t) + z_1(t)z_2(t)$$

is a sum of product.

- In a similar way

$$\vec{\gamma_1} \times \vec{\gamma_2} = \det \begin{bmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{e_1} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{e_2} + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{e_3}$$

$$= \begin{bmatrix} x_1 y_2 - y_1 x_2 \\ z_1 x_2 - x_1 z_2 \\ y_1 z_2 - z_1 y_2 \end{bmatrix}$$

has also every coordinate as a sum of products.

→ A last product : the scalar multiplication.

There again we have :

$$\frac{d}{dx}\left(f(x) \cdot \vec{\gamma}(x)\right)$$

$$= \left(\frac{df}{dx}\right) \cdot \vec{\gamma}(x) + f(x) \cdot \left(\frac{d\vec{\gamma}}{dx}\right).$$

DERIVATION OF A QUOTIENT :

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\left(\frac{df}{dx}\right) \cdot g - f \cdot \left(\frac{dg}{dx}\right)}{g^2}$$

→ we do not know ~~how~~ to divide by a vector
but

$$\frac{d}{dx}\left(\frac{\vec{\gamma}(x)}{g(x)}\right) = \text{the same expression.}$$

THE CHAIN RULE (VERSION 1)

Consider $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$
$x \mapsto \vec{\gamma}(x)$

and $f : \mathbb{R} \to \mathbb{R}$
$t \mapsto f(t)$

We want to compute the derivative of

$\vec{\gamma} \circ f : \mathbb{R} \to \mathbb{R}^3$
$t \mapsto \vec{\gamma}(f(t))$

Interpretation: f could be a "change of parametrization",

for example $f : t \mapsto 2t$ for a fly that goes twice as fast.

→ A naive idea would be to write:

$$\frac{d}{dt}\left(\vec{\gamma}(2t)\right)\Big|_{t_0} = \frac{d\vec{\gamma}}{dt}\Big|_{2t_0} \quad \text{but it is wrong!}$$

"When the fly goes twice as fast, the speed vector must be multiplied by 2".

$$\frac{d}{dt}\left(\vec{\gamma}(2t)\right)\Big|_{t_0} = \frac{d\vec{\gamma}}{dt}\Big|_{2t_0} \times 2 .$$

In general we have:

CHAIN RULE (VERSION 1):

$$\boxed{\frac{d}{dt}\left(\vec{\gamma}(f(t))\right)\Big|_{t_0} = \frac{d\vec{\gamma}}{dx}\Big|_{f(t_0)} \cdot \frac{df}{dt}\Big|_{t_0}}$$

Idea of the proof:

$$\frac{d}{dt}\left(\vec{\gamma}(f(t))\right)\Big|_{t_0} = \lim_{\Delta t \to 0} \frac{\vec{\gamma}(f(t_0+\Delta t)) - \vec{\gamma}(f(t_0))}{\Delta t}$$

and

$$\frac{\vec{\gamma}(f(t_0+\Delta t)) - \vec{\gamma}(f(t_0))}{\Delta t}$$

$$= \frac{\vec{\gamma}(f(t_0+\Delta t)) - \vec{\gamma}(f(t_0))}{f(t_0+\Delta t) - f(t_0)} \cdot \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{d\vec{\gamma}}{dx}\Big|_{f(t_0)} \qquad\qquad \frac{df}{dt}\Big|_{t_0}$$

- An abreviated notation:

$$\frac{d\vec{\gamma}}{dt} = \frac{d\vec{\gamma}}{dx} \cdot \frac{dx}{dt}$$

(with $x = f(t)$, and with the risk of forgetting at which points are taken the derivatives!)

$$\left.\frac{d\vec{\gamma}}{dt}\right|_{t_0} = \left.\frac{d\vec{\gamma}}{dx}\right|_{f(t_0)} \cdot \left.\frac{dx}{dt}\right|_{t_0}$$

- In Lagrange notation:

$$(\vec{\gamma} \circ f)' = (\vec{\gamma}\,' \circ f) \cdot f'$$

WITH VARIABLES

$$\left(\vec{\gamma}(f(t))\right)' = \vec{\gamma}\,'(f(t)) \cdot f'(t).$$

Examples:

① Compute $\frac{d}{dx}\left(e^{(\cos(x))^2}\right)$.

$$\left.\frac{d}{dx}\left(e^{(\cos(x))^2}\right)\right|_x = \left.\frac{d\exp}{dx}\right|_{(\cos(x))^2} \cdot \left.\frac{d}{dx}\left(\cos(x)^2\right)\right|_x$$

$$= \exp\left((\cos(x))^2\right) \cdot \left.\frac{df}{dx}\right|_{\cos(x)} \cdot \left.\frac{d}{dx}\cos(x)\right|_x ,$$

with $f : x \mapsto x^2$

$$= \exp\left((\cos(x))^2\right) \cdot 2\cos(x) \cdot (-\sin(x))$$

use $f'(x) = 2x$

$$= -2\, e^{(\cos(x))^2} \cdot \cos(x)\sin(x)$$

(2) derivative of something like $f(x)^{g(x)}$.

→ we ~~use~~ use $\boxed{x^y = e^{y\ln(x)}}$

(definition of x^y).

→ example:

$$\frac{d}{dx}\left(\cos(x)^{\sin(x)}\right) = \frac{d}{dx}\left(e^{\sin(x)\ln(\cos(x))}\right)$$

$$= e^{\sin(x)\ln(\cos(x))} \cdot \underbrace{\left[\cos(x)\ln(\cos(x)) + \sin(x)\frac{1}{\cos(x)}\cdot(-\sin x)\right]}_{\text{derivative of a product } \sin(x)\ln(\cos(x))}$$

$$= \cos(x)^{\sin(x)} \cdot \left[\cos(x)\ln(\cos(x)) - \frac{\sin^2(x)}{\cos(x)}\right].$$