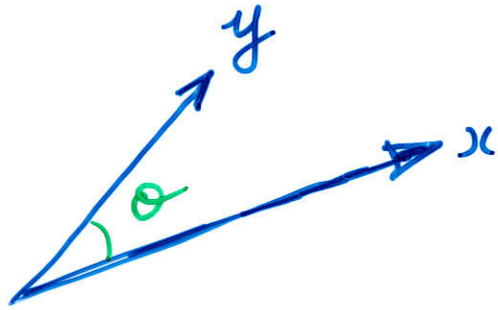
[4] THE CAUCHY-SCHWARZ INEQUALITY

From the formula

$$\langle x, y \rangle = \|x\| \, \|y\| \cos\theta$$



it follows that

absolute value

$$|\langle x, y \rangle| = \|x\| \, \|y\| \underbrace{|\cos\theta|}_{\leq 1}$$

with equality $\iff \theta = 0$ or π

This is called the Cauchy-Schwarz inequality

Thus, for any vectors x and y we have that

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

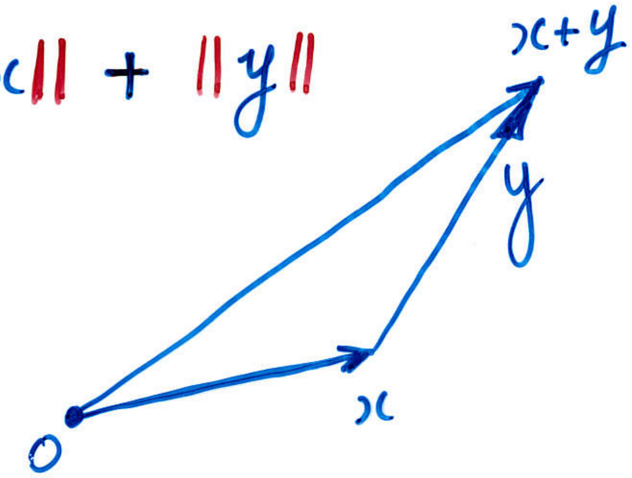
with equality

$\iff$ $\{$ x and y lie on the SAME line through the origin

[5] THE TRIANGLE INEQUALITY

For any vectors x and y

$$\|x+y\| \leq \|x\| + \|y\|$$



with equality

$\iff$ x and y lie in the same direction along the same line through the origin

PROOF:

$$0 \leq \|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

with equality $\iff \langle x,y\rangle \geq 0$ acute angle

$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

By Cauchy-Schwarz with equality $\iff$ .... same line

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Now, take $\sqrt{\ }$ of both sides and note that we get equality $\iff$ Both inequalities hold

QED.

[6] The Reverse Triangle Inequality

For any vectors x and y we have that

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\|$$

PROOF:

$$\|x-y\|^2 = \langle x-y, x-y \rangle$$

$$= \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

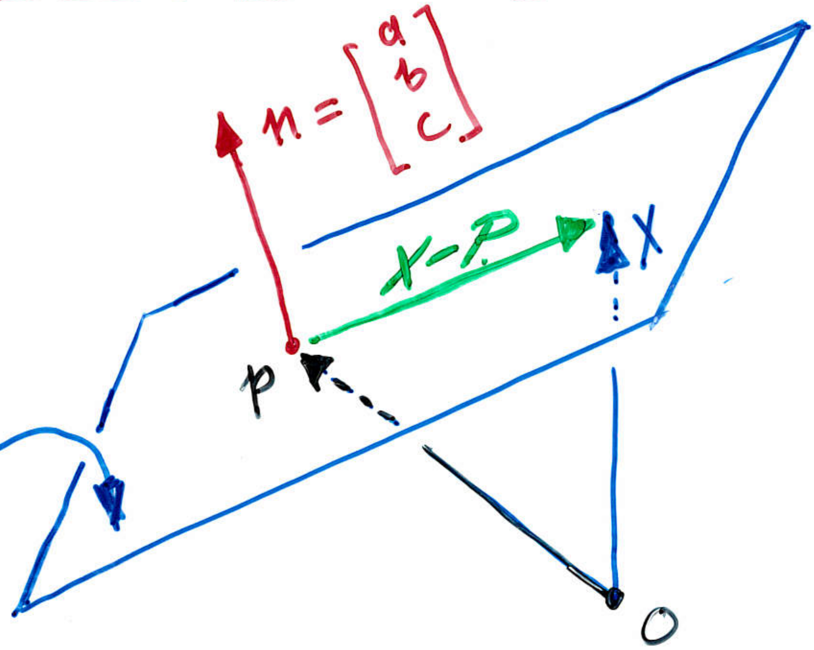
$$\vdots$$

$$\geq (\|x\| - \|y\|)^2 \geq 0$$

You fill in these steps as before, just be careful about the direction of inequalities

What this says is that if x is near y, that is, if $\|x-y\|$ is small, then $\|x\|$ is near $\|y\|$. That is $\big|\,\|x\|-\|y\|\,\big|$ is small. In other words, the function $\|\ \ \| : \mathbb{R}^3 \to [0,\infty) : x \mapsto \|x\|$ is CONTINUOUS.

[7] THE EQUATION OF A PLANE IN $\mathbb{R}^3$



The plane through the point "p" having the vector $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a NORMAL

Clearly

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ lies on this plane $\iff (X-p) \perp n$

$\iff \langle X-p, n \rangle = 0$

$\iff \langle X, n \rangle - \langle p, n \rangle = 0$

$$\Longleftrightarrow \boxed{\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\rangle = \underbrace{\langle p, n \rangle}}$$

Once p and n are given, this is just a constant which we denote by "d"

$$\Longleftrightarrow \boxed{ax + by + cz = d}$$

EXAMPLE: The plane passing through $p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ having $n = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ as normal is given by

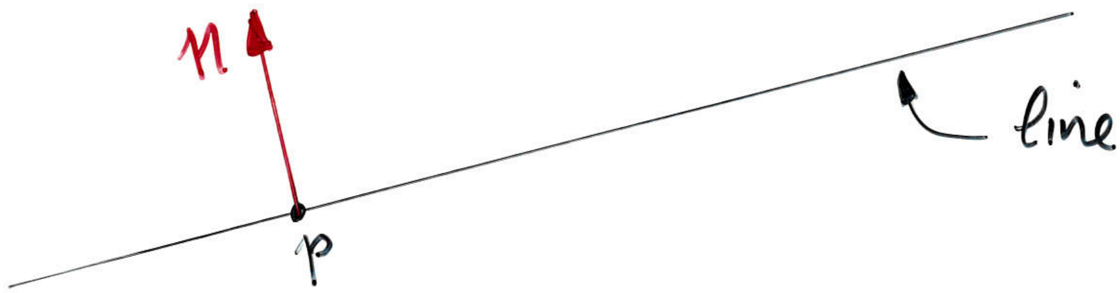
$$2x + 4y - z = \langle p, n \rangle$$
$$= \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\rangle = 7.$$

Lines in Space

We know that in the xy-plane, the equation

$$ax + by = c$$

represents a line. That is, it's solution set is a line having $n = \begin{bmatrix} a \\ b \end{bmatrix}$ as a NORMAL VECTOR.



Two such lines (unless they are parallel) intersect in a point. For example:

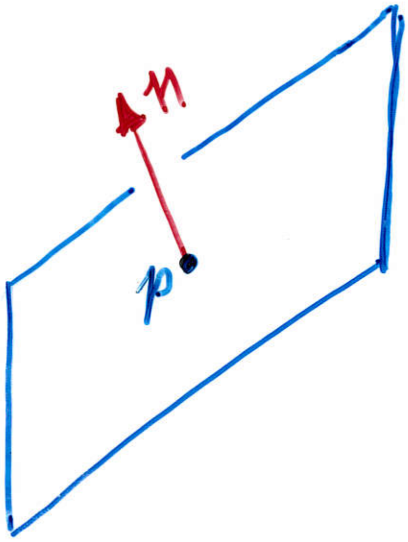
$$\boxed{\begin{aligned} 1x + 2y &= 3 \\ 2x + 3y &= 1 \end{aligned}}$$

If we want to "find this point" we must "solve these equations".

Similarly (as we've seen) the equation

$$ax + by + cz = d$$

has as solution set a plane in space



with $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as NORMAL vector. Two such planes (unless they are parallel) intersect in a line.

To "find the line of intersection" of such planes, for example,

$$\boxed{\begin{aligned} 1x + 2y + 4z &= 2 \\ 2x + 3y - 1z &= 1 \end{aligned}}$$

we must "solve these equations". To do this THERE IS A STANDARD PROCEDURE which YOU are expected to follow VERBATIM.

We illustrate this standard procedure by the example just given:

$$\boxed{\begin{aligned} 1x + 2y + 4z &= 2 \\ 2x + 3y - 1z &= 1 \end{aligned}}$$

$$\Big\downarrow \quad \begin{aligned} R_1 &\longrightarrow R_1 \\ R_2 &\longrightarrow R_2 - 2R_1 \end{aligned}$$

$$\boxed{\begin{aligned} 1x + 2y + 4z &= 2 \\ -1y - 9z &= -3 \end{aligned}}$$

$$\Big\downarrow \quad \begin{aligned} R_1 &\longrightarrow R_1 \\ R_2 &\longrightarrow -R_2 \end{aligned}$$

$$\boxed{\begin{aligned} 1x + 2y + 4z &= 2 \\ 1y + 9z &= 3 \end{aligned}}$$

$$\Big\downarrow \quad \begin{aligned} R_1 &\longrightarrow R_1 - 2R_2 \\ R_2 &\longrightarrow R_2 \end{aligned}$$

$$\boxed{\begin{aligned} 1x \quad - 14z &= -4 \\ 1y + 9z &= 3 \end{aligned}}$$

$$\Big\downarrow$$

$$\boxed{\begin{aligned} 1x &= -4 + 14z \\ 1y &= 3 - 9z \end{aligned}}$$

$$\boxed{\begin{aligned} x &= -4 + 14z \\ y &= 3 - 9z \\ z &= 0 + 1z \end{aligned}}$$

The silly equation

Thus we have represented our line (that is, the **SOLUTION SET** of the simultaneous equations) by a map

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 : z \longmapsto \gamma(z) = \underbrace{\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}}_{p} + z \underbrace{\begin{bmatrix} 14 \\ -9 \\ 1 \end{bmatrix}}_{v}$$

This is the line

through $p = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ in the DIRECTION $v = \begin{bmatrix} 14 \\ -9 \\ 1 \end{bmatrix}$