Chapter 3: Calculus of functions of several variables

GENERALITIES:

In the previous chapter, we've been looking at functions $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$ (parametrized curves).

Here we consider a dual object,

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

Motivation / mental representation of f:

Suppose we measure temperature at every point in space,
The associated mathematical object is naturally

$$f : \mathbb{R}^3 \to \mathbb{R}$$
$$\vec{x} = (x, y, z) \longmapsto \text{the temperature at } \vec{x} = (x, y, z).$$

LIMITS, CONTINUITY:

Our definition of the limit adapts nicely to the new context of $f : \mathbb{R}^3 \to \mathbb{R}$.

$$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = \ell$$

$\iff$ For every $\varepsilon > 0$, there exists $\delta > 0$

$\|(x,y,z) - (x_0,y_0,z_0)\| < \delta \Rightarrow |f(t) - \ell| < \varepsilon$.

REMARK:

→ We just had to consider the proper distance on $\mathbb{R}^3$,

$$d\big((x,y,z), (x_0,y_0,z_0)\big)$$

$$= \|(x,y,z) - (x_0,y_0,z_0)\| = \left\| \begin{matrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{matrix} \right\|$$

$$= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

(from Chapter 1).

DEFINITION (The USUAL)

We say that $f: \mathbb{R}^3 \to \mathbb{R}$ is continuous at $(x,y,z) = (x_0, y_0, z_0)$ if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0).$$

Example: $f: \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \mapsto x^2 y + 3y + 4$$

is continuous at $(x_0, y_0, z_0) = (0,0,0)$.

$$\lim_{(0,0,0)} f = 4 = f(4).$$

CAREFUL THOUGH, FOR THE NIGHT (and the realm of functions of several variables) IS DARK AND FULL OF TERRORS!

→ The definition of the limit is STRONG, and it is easy to mistakenly believe a function is continuous / has a limit, etc... when it is NOT.

Example :

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0). \end{cases}$$

$$\rightarrow \quad \lim_{x \to 0} f(x,0) = 0$$

$$\lim_{y \to 0} f(0,y) = 0$$

$$\lim_{x \to 0} (x, mx) = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 x^2} = \frac{m \cancel{x^2}}{(1+m^2)\cancel{x^2}} = \frac{m}{1+m^2}$$

SEE TUTORIAL 4.

Worse : There are examples of functions that are continuous along any line going through $\vec{0}$, but not continuous along other curves going through $\vec{0}$!

Conclusion : → We really can't say anything from a function's behaviour along trajectories.

→ As usual, we shall stay away from the "bad-behaving" functions in this class. Please remember that they still exist.

A LATE REMARK: We've used the notations

$\vec{x}$, (x,y,z) , $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a vector in $\in \mathbb{R}^3$.

I hope you are not shocked.

PARTIAL DERIVATIVES

It is difficult to talk about a speed without having a "one-dimensional evolution."

$f: \mathbb{R} \to \mathbb{R}$ $\qquad \vec{\gamma}: \mathbb{R} \to \mathbb{R}^3$

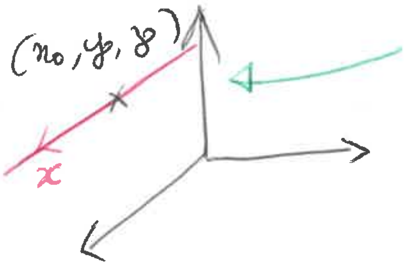
are nice because $t \in \mathbb{R}$ can be seen as time.

We are not impressed and shall chose a direction in $\mathbb{R}^3$ in which we differentiate.

Let $\begin{aligned} f: \mathbb{R}^3 &\to \mathbb{R} \\ (x,y,z) &\mapsto f(x,y,z) \end{aligned}$.

If we consider (x_0,y_0,z_0) and fix $\begin{cases} y = y_0, \\ z = z_0 \end{cases}$,

we obtain $\begin{aligned} g: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x,y_0,z_0) \end{aligned}$



We are looking at the "temperature" $f(x,y,z)$ along the red line to obtain a function $g: \mathbb{R} \to \mathbb{R}$.

And of course we differentiate g.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)} = g'(x_0) = \left.\frac{dg}{dx}\right|_{x=x_0}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

Is the partial derivative of f with respect to the coordinate x.

REMARK:
- observe that in the definition $y = y_0$ and $z = z_0$ are absolutely fixed.
- However once the limit is taken, we allow ourselves to change (x_0, y_0, z_0), so that what we've defined is a function of $(x_0, y_0, z_0) \in \mathbb{R}^3$.

$$\frac{\partial f}{\partial x} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x_0, y_0, z_0) \longmapsto \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)}.$$

→ Recipe: At point (x_0, y_0, z_0), fix a "red line" in direction x, and compute the derivative of the function $g : \mathbb{R} \to \mathbb{R}$ that is f along the red line.

→ Clearly that recipe can be applied at every $(x_0, y_0, z_0) \in \mathbb{R}^3$.

→ We can drop the 0s carefully and write:

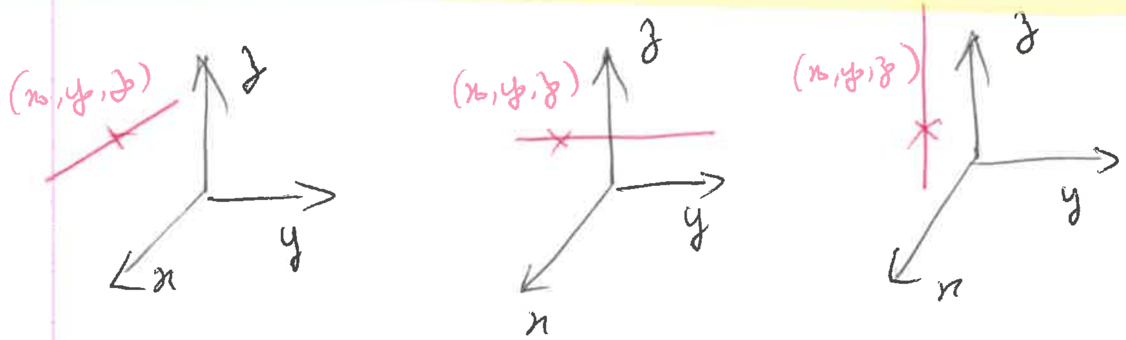
$$\frac{\partial f}{\partial x} : \mathbb{R}^3 \to \mathbb{R}$$
$$(x, y, z) \longmapsto \left.\frac{\partial f}{\partial x}\right|_{(x, y, z)}.$$

OF COURSE we also define:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y, z_0) - f(x_0,y_0,z_0)}{\Delta y}$$

and

$$\left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} = \lim_{\Delta z \to 0} \frac{f(x_0, y_0, z_0+\Delta z) - f(x_0,y_0,z_0)}{\Delta z}$$



↳ The three "<u>red lines</u>" at (x_0,y_0,z_0), that allow the definition of $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)}$, $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)}$

and $\left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)}$.

Examples: Practically we "differentiate along a coordinate" by "just differentiating and keeping in mind what is constant and what is not".

$$\frac{\partial}{\partial x}(x+y) = 1$$

$$\frac{\partial}{\partial x}\left(x^2+\cos(y)\right) = 2x .$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$
$$(x,y,z) \mapsto x\cos(y+z^2)$$