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CROSS DERIVATIVES:

Now that we have

$$\frac{\partial f}{\partial x} : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \mapsto \left. \frac{\partial f}{\partial x} \right|_{(x, y, z)},$$

we are allowed to consider

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) : \mathbb{R}^3 \rightarrow \mathbb{R},$$

and we denote this new function by

$$\boxed{\frac{\partial^2 f}{\partial y \partial x}}$$

← Cross-derivative
with respect to x and y .

In a similar way we consider

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \text{ denoted by } \boxed{\frac{\partial^2 f}{\partial x^2}}.$$

(double derivative with respect to x)

THEOREM: For $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ a
"nice function", we have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Remarks: For "nice enough" functions,

we can consider any combination like

$$\frac{\partial^3 f}{\partial x \partial y \partial z}, \text{ and the order in which we differentiate doesn't matter.}$$

E.g.:

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial x^2 \partial y}$$

DIRECTIONAL DERIVATIVES

So far we have been considering derivatives in very specific directions, parallel to the x, y, z axis.

More generally for every $\vec{u} \neq \vec{0}$ we define

$$\left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{x}_0} = \lim_{\Delta t \rightarrow 0} \frac{f(\vec{x}_0 + \Delta t \vec{u}) - f(\vec{x}_0)}{\Delta t},$$

the directional derivative at \vec{x}_0 along the vector \vec{u} .

REMARK: \rightarrow We are fixing a line that follows

the direction of \vec{u} at point \vec{x}_0 , looking at f "on this line only" to obtain a function $g: \mathbb{R} \rightarrow \mathbb{R}$ and we differentiate.

* Scaling: Consider $\vec{u} \neq \vec{0}$ and $\lambda \neq 0$.

We have

$$\left. \frac{\partial f}{\partial (\lambda \vec{u})} \right|_{\vec{x}_0} = \lim_{\Delta t \rightarrow 0} \frac{f(\vec{x}_0 + \Delta t \lambda \vec{u}) - f(\vec{x}_0)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{\lambda [f(\vec{x}_0 + \lambda \Delta t \vec{u}) - f(\vec{x}_0)]}{\lambda \Delta t}$$

$$= \lambda \lim_{\Delta t \rightarrow 0} \frac{f(\vec{x}_0 + \lambda \Delta t \vec{u}) - f(\vec{x}_0)}{\lambda \Delta t}$$

$$= \lambda \lim_{\Delta t \rightarrow 0} \frac{f(\vec{x}_0 + \Delta t \vec{u}) - f(\vec{x}_0)}{\Delta t}$$

$$= \lambda \left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{x}_0}$$

↳ Scaling \vec{u} by λ also scales the "speed"

$\frac{\partial f}{\partial \vec{u}}$, which makes sense.

REMARK: ⊗ The notation $\frac{\partial f}{\partial \vec{u}}$ with \vec{u} in the bottom

is a bit weird: it is just a notation,

by no mean are we dividing by $\partial \vec{u}$
here, or \vec{u} .

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⊗ We will often consider $\frac{\partial f}{\partial \vec{u}}$ for $\|\vec{u}\|=1$,

because it is enough, and natural.

In this case we talk about the derivative in
the direction \vec{u} .

THEOREM: For $\vec{u} = \begin{bmatrix} x_{\vec{u}} \\ y_{\vec{u}} \\ z_{\vec{u}} \end{bmatrix} \neq \vec{0}$

and $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ a "nice" function,

$$\frac{\partial f}{\partial \vec{u}} \Big|_{\vec{x}_0} = x_{\vec{u}} \frac{\partial f}{\partial x} \Big|_{\vec{x}_0} + y_{\vec{u}} \frac{\partial f}{\partial y} \Big|_{\vec{x}_0} + z_{\vec{u}} \frac{\partial f}{\partial z} \Big|_{\vec{x}_0}$$

REMARKS:

⊗ This is the formula we use to compute a directional derivative!

⊗ In matrix notation:

$$\left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{x}_0} = \left[\left. \frac{\partial f}{\partial x}}, \left. \frac{\partial f}{\partial y}}, \left. \frac{\partial f}{\partial z} \right] \right|_{\vec{x}_0} \cdot \begin{bmatrix} x_{\vec{u}} \\ y_{\vec{u}} \\ z_{\vec{u}} \end{bmatrix}.$$

This comes from the fact that

$$\left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{x}_0} : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$\vec{u} \mapsto \left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{x}_0}$$

⚠ Here \vec{x}_0 is fixed and \vec{u} is the variable!

is a linear mapping.