BACK TO CHAPTER 3

We are now equipped to state another version of the chain rule.

- In our first version we had

  $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$

  and $f : \mathbb{R} \to \mathbb{R}$, and we obtained

$$\boxed{\frac{d}{dt} \vec{\gamma} \circ f \Big|_{t_0} = \frac{d\vec{\gamma}}{dt}\Big|_{f(t_0)} \cdot \frac{df}{dt}\Big|_{t_0}}$$

- This time we consider

  $f : \mathbb{R}^3 \to \mathbb{R}$

  and $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^3$, and we want to express $\frac{d}{dt} f \circ \vec{\gamma} \Big|_{t_0}$

Interpretation:

If $f: \mathbb{R}^3 \to \mathbb{R}$ is the "temperature in the room"

and $\vec{\gamma}: \mathbb{R} \to \mathbb{R}^3$ is the "trajectory of a fly",

$f \circ \vec{\gamma}: \mathbb{R} \to \mathbb{R}$ is the "temperature along the trajectory of a fly".

THEOREM (Chain rule version 2)

If we write $\vec{\gamma}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

$$\boxed{\begin{aligned} &\left.\frac{d f \circ \vec{\gamma}}{dt}\right|_{t_0} = \\ &\left.\frac{\partial f}{\partial x}\right|_{\vec{\gamma}(t_0)} \cdot \left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial y}\right|_{\vec{\gamma}(t_0)} \cdot \left.\frac{dy}{dt}\right|_{t_0} + \left.\frac{\partial f}{\partial z}\right|_{\vec{\gamma}(t_0)} \cdot \left.\frac{dz}{dt}\right|_{t_0} \end{aligned}}$$

In "short" (be careful):

$$\boxed{\frac{d\, f\circ\vec{\gamma}}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}}$$

REMARK: This is exactly the partial derivative of f along the vector $\frac{d\vec{\gamma}}{dt}$!
(which makes sense).

IN MATRIX NOTATION:

$$\frac{d\, f\circ\vec{\gamma}}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$