A Brief Note on Determinants

Motivation: To solve the equations

$$\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}$$

Proceed as follows

$$\Updownarrow$$

$$\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \\ adx + bdy &= dk_1 \\ bcx + bdy &= bk_2 \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \\ (ad-bc)x &= (dk_1 - bk_2) \end{aligned}$$

Thus, for a given k_1, k_2 the original system of equations

$$\boxed{\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}}$$

$\boxed{\text{has a UNIQUE SOLUTION } (x, y)} \iff \boxed{(ad - bc) \neq 0}$

The quantity $(ad - bc)$ is called the DETERMINANT of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A similar situation arises in the case of three linear equations in three unknowns; x, y, z.

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= \alpha \\ b_1 x + b_2 y + b_3 z &= \beta \\ c_1 x + c_2 y + c_3 z &= \gamma . \end{aligned}$$

Here we consider the

DETERMINANT of the coefficient matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ which is defined by

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= \boxed{a_1 \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}}$$

$$= \boxed{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)}$$

Again the system of equations has a <u>UNIQUE SOLUTION</u> $\iff \det\begin{pmatrix}\text{coefficient} \\ \text{matrix}\end{pmatrix} \neq 0$.

THE CROSS PRODUCT:

given any two vectors

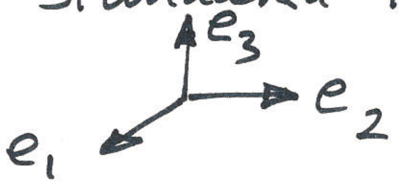
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

we denote their CROSS PRODUCT by $(x \times y)$ and define it to be

$$x \times y = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

The FORMAL DETERMINANT As defined on page 3

Here $e_1\ e_2\ e_3$ is the standard FRAME



$$= (x_2 y_3 - x_3 y_2) e_1 + (x_3 y_1 - x_1 y_3) e_2 + (x_1 y_2 - x_2 y_1) e_3$$

That is

$$x \times y = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

OBSERVE that

$$\langle x \times y, z \rangle = \det \begin{bmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

and, in particular, that

$$\langle (x \times y), x \rangle = 0$$

$$\langle (x \times y), y \rangle = 0.$$

Thus

$(x \times y)$ is PERPENDICULAR to x and to y.

From the formula

$$x \times y = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

we have that

$$\| x \times y \|^2 = (x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2 + (x_1 y_2 - x_2 y_1)^2$$

compute and compute
$$\left\{ \begin{aligned} &= \cdots \\ &\vdots \\ &= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1 y_1 + x_2 y_2 + x_3 y_3)^2 \end{aligned} \right.$$

$$= \|x\|^2 \|y\|^2 - \langle x, y \rangle^2$$

$$= \|x\|^2 \|y\|^2 - \|x\|^2 \|y\|^2 \cos^2\theta$$

$$= \|x\|^2 \|y\|^2 (1 - \cos^2\theta)$$

$$= \|x\|^2 \|y\|^2 \sin^2\theta$$

$$\Rightarrow \|x \times y\| = \|x\| \|y\| \sin\theta$$

Thus we have that

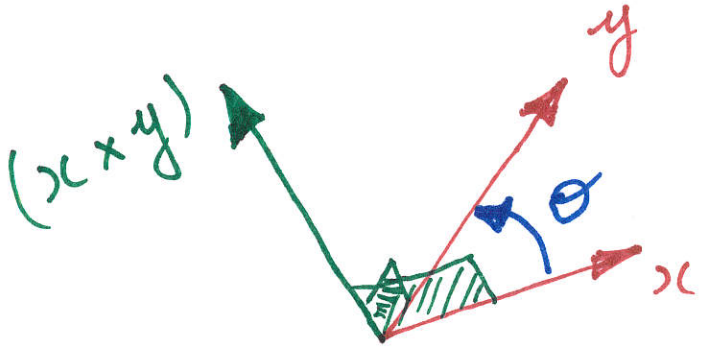
$(x \times y)$ is $\perp$ to Both x & y

and $\|x \times y\| = \|x\| \|y\| \sin\theta$.

It turns out that the THREE VECTORS

$$x \; , \; y \; , \; (x \times y)$$

taken in that order form a right handed system so that the correct picture is



EXAMPLE: Calculate $x \times y$

where $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

ANSWER

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

These lines mean that you take the formal determinant

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} e_1 - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} e_2 + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} e_3$$

$$= (12 - 15) e_1 - (6 - 12) e_2 + (5 - 8) e_3$$

$$= -3 e_1 + 6 e_2 - 3 e_3$$

$$= \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}.$$

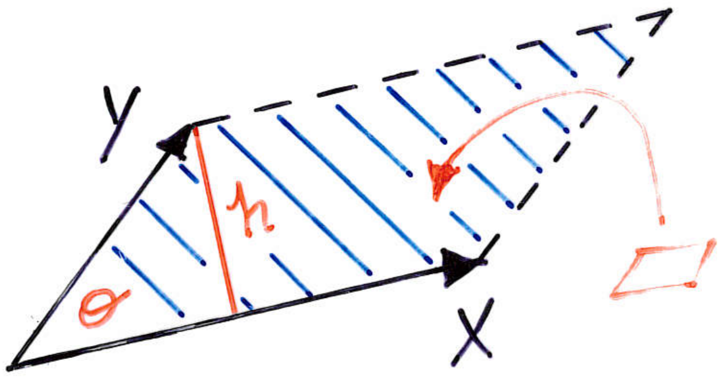
APPLICATIONS OF THE CROSSPRODUCT

[1] AREA OF A PARALLELOGRAM

Let ▱ denote the parallelogram in $\mathbb{R}^3$ which is spanned by the vectors X and Y, then

$$\text{area}(▱) = \|X \times Y\|$$

PROOF:



Clearly

$$\sin\theta = \frac{h}{\|Y\|}$$

$\Rightarrow$ $h = \|Y\| \sin\theta$. Therefore

$$\text{area}(▱) = \|X\|\, h = \|X\|\,\|Y\| \sin\theta$$
$$= \|X \times Y\|$$

[2] AREA OF A SHADOW

Let ▱ be a parallelogram in $\mathbb{R}^3$ with UNIT NORMAL VECTOR n and let $\mathcal{P}$ be the plane through the origin in $\mathbb{R}^3$ with UNIT NORMAL ξ.

When light from infinity shining parallel to ξ falls on ▱ it casts a shadow $P(▱)$ on the plane $\mathcal{P}$.

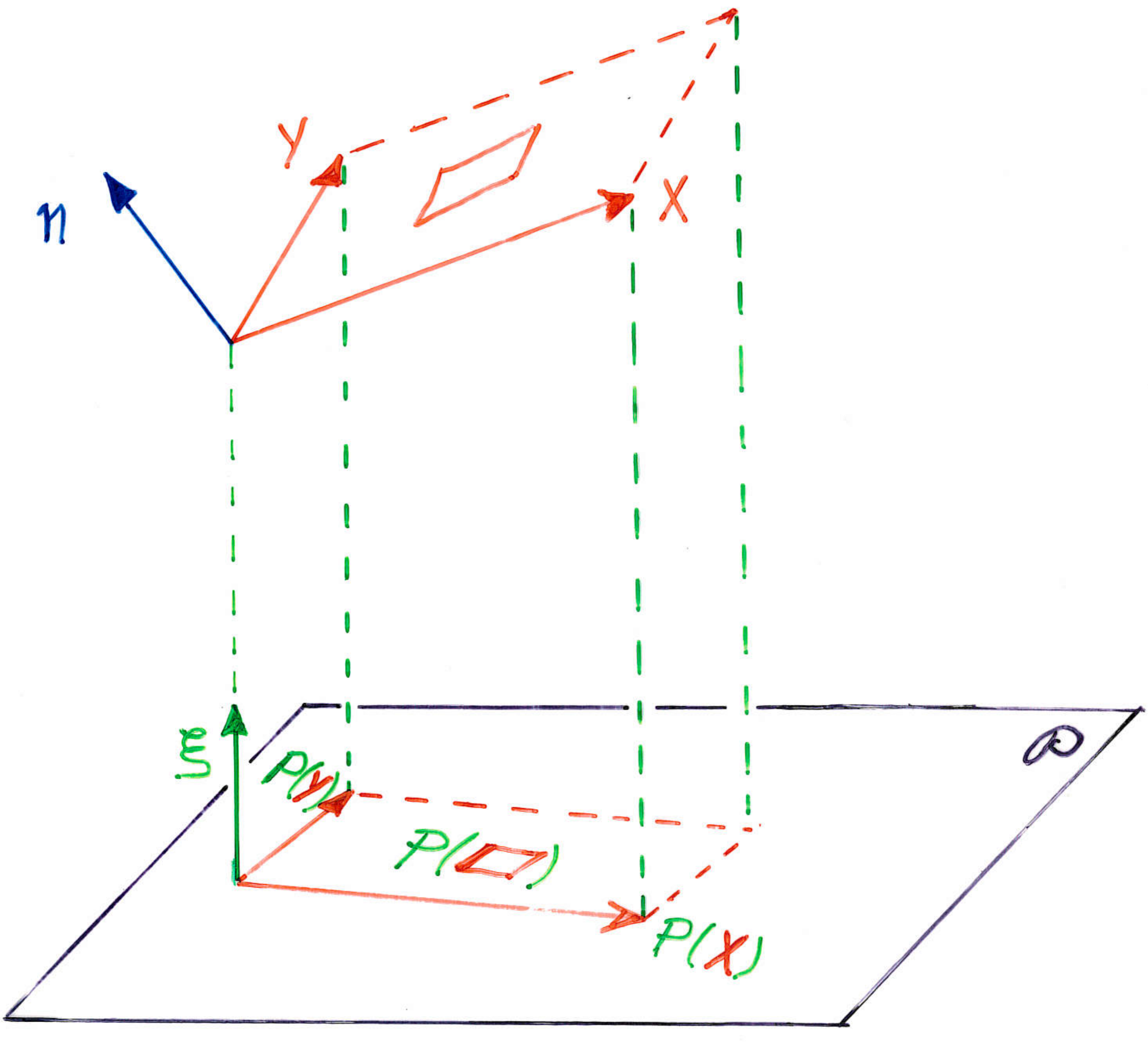
The areas of ▱ and $P(▱)$ are related by

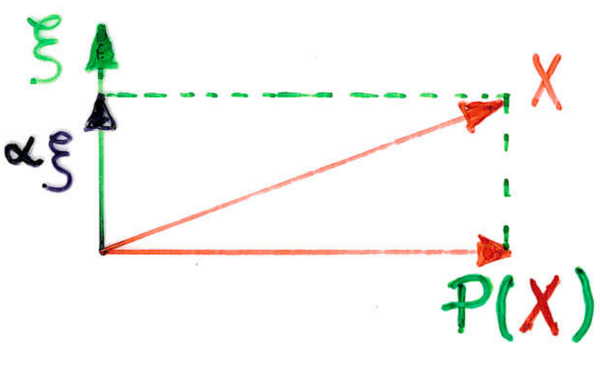
$$\text{area}\, P(▱) = (\text{area}\, ▱)\,\left|\langle n, \xi \rangle\right|$$

ABSOLUTE VALUE

PROOF: as in application [1]

let ▱ be spanned by the vectors X and Y





clearly

$$X = P(X) + \alpha\xi$$

where $\alpha = \langle X, \xi \rangle$

Choose (the orientation of) n and ξ so that both triples

$$(X, Y, n)$$

and

$$(P(X), P(Y), \xi)$$

are RIGHT-HANDED. Then

$$n = \frac{X \times Y}{\|X \times Y\|}$$

and

$$\xi = \frac{P(X) \times P(Y)}{\|P(X) \times P(Y)\|}$$

In particular

$$\boxed{\begin{array}{l} X \times Y = \|X \times Y\| \, n \\ \text{and} \\ P(X) \times P(Y) = \|P(X) \times P(Y)\| \, \xi \end{array}} \quad (*)$$

Now,

$$\text{area}\, P(\square) = \| P(X) \times P(Y) \|$$

$$\overset{\text{by }(*)\text{ page }4}{=} \langle P(X) \times P(Y) \,,\, \xi \rangle$$

$$= \langle (X - \alpha \xi) \times (Y - \beta \xi) \,,\, \xi \rangle$$

$$= \langle [X \times Y - \beta X \times \xi - \alpha \xi \times Y + 0] \,,\, \xi \rangle$$

$$= \langle X \times Y \,,\, \xi \rangle + 0 + 0$$

$$\overset{\text{by }(*)\text{ page }4}{=} \langle \| X \times Y \| n \,,\, \xi \rangle$$

$$= \| X \times Y \| \langle n \,,\, \xi \rangle$$

$$= \text{area}(\square) \langle n \,,\, \xi \rangle$$

QED.