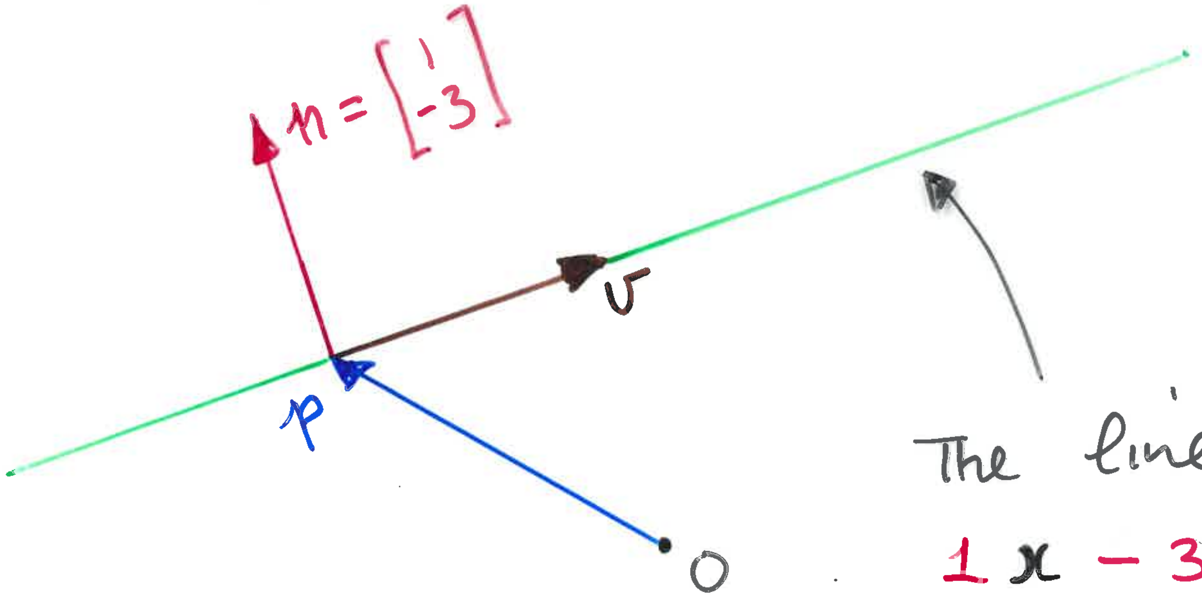
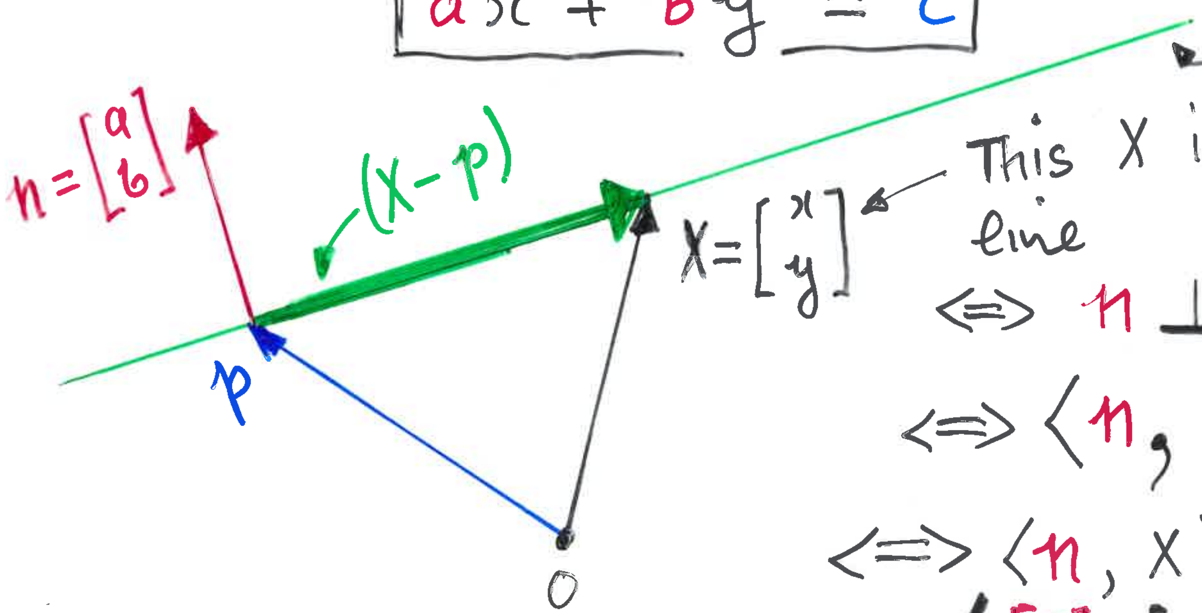


The line

$1x - 3y = 4$

REMARK: In the case of the line, we could have worked as we did in the case of the plane and SHOW that the line passing through $p \in \mathbb{R}^2$ and having $n = \begin{bmatrix} a \\ b \end{bmatrix}$ as normal has equation

$$\boxed{ax + by = c}$$



This X is on this line

$\iff n \perp (X-p)$

$\iff \langle n, X-p \rangle = 0$

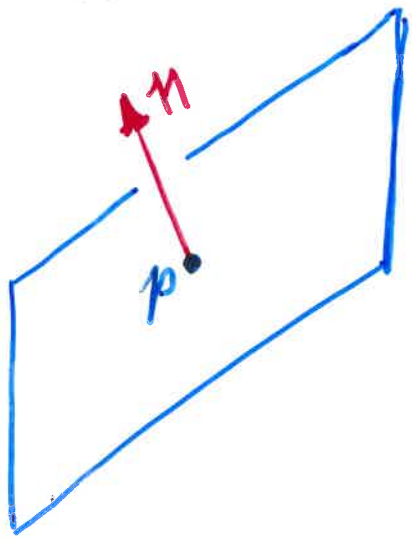
$\iff \langle n, X \rangle = \langle n, p \rangle$

$\iff \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \overset{\shortparallel}{c}$

As we have seen, the equation

$$ax + by + cz = d$$

has as solution set a plane in space



with $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as NORMAL vector. Two such planes (unless they are parallel) intersect in a line.

To "find the line of intersection" of such planes, for example,

$$\boxed{\begin{array}{rcl} 1x + 2y + 4z &=& 2 \\ 2x + 3y - 1z &=& 1 \end{array}}$$

we must "solve these equations". To do this THERE IS A STANDARD PROCEDURE which YOU are expected to follow VERBATIM.

We illustrate this standard procedure by the example just given:

$$\boxed{\begin{array}{rcrcrcl} 1x & + & 2y & + & 4z & = & 2 \\ 2x & + & 3y & - & 1z & = & 1 \end{array}}$$

$$\Big\downarrow \quad \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 - 2R_1 \end{array}$$

$$\boxed{\begin{array}{rcrcrcl} 1x & + & 2y & + & 4z & = & 2 \\ & & -1y & - & 9z & = & -3 \end{array}}$$

$$\Big\downarrow \quad \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow -R_2 \end{array}$$

$$\boxed{\begin{array}{rcrcrcl} 1x & + & 2y & + & 4z & = & 2 \\ & & 1y & + & 9z & = & 3 \end{array}}$$

$$\Big\downarrow \quad \begin{array}{l} R_1 \longrightarrow R_1 - 2R_2 \\ R_2 \longrightarrow R_2 \end{array}$$

$$\boxed{\begin{array}{rcrcrcl} 1x & & & - & 14z & = & -4 \\ & & 1y & + & 9z & = & 3 \end{array}}$$

$$\Big\downarrow$$

$$\boxed{\begin{array}{rcl} 1x & = & -4 + 14z \\ 1y & = & 3 - 9z \end{array}}$$

$$\boxed{\begin{aligned} x &= -4 + 14z \\ y &= 3 - 9z \\ z &= 0 + 1z \end{aligned}}$$

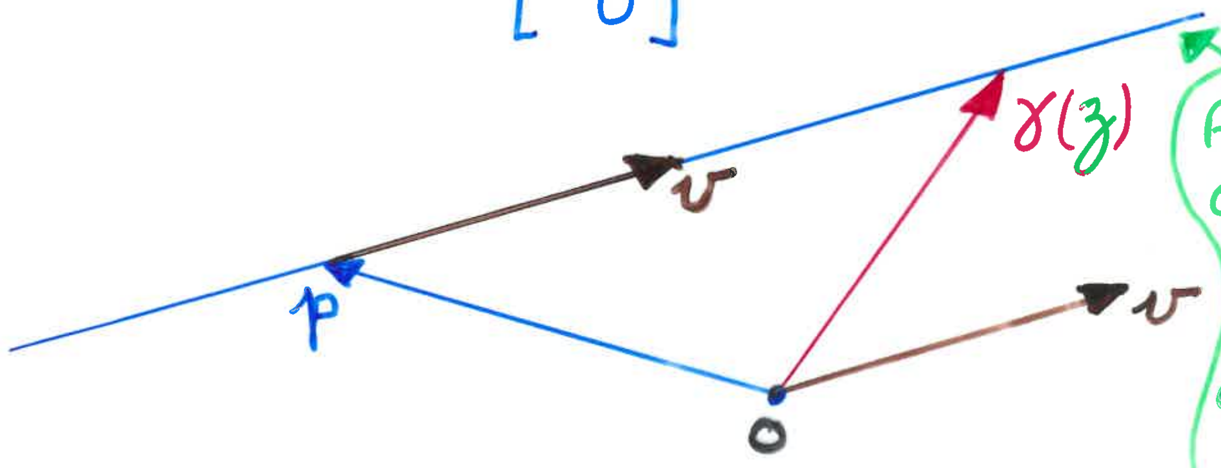
The silly equation

Thus we have represented our line (that is, the **SOLUTION SET** of the simultaneous equations) by a map

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 : z \longmapsto \gamma(z) = \underbrace{\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}}_{p} + z \underbrace{\begin{bmatrix} 14 \\ -9 \\ 1 \end{bmatrix}}_{v}$$

This is the line

through $p = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ in the DIRECTION $v = \begin{bmatrix} 14 \\ -9 \\ 1 \end{bmatrix}$

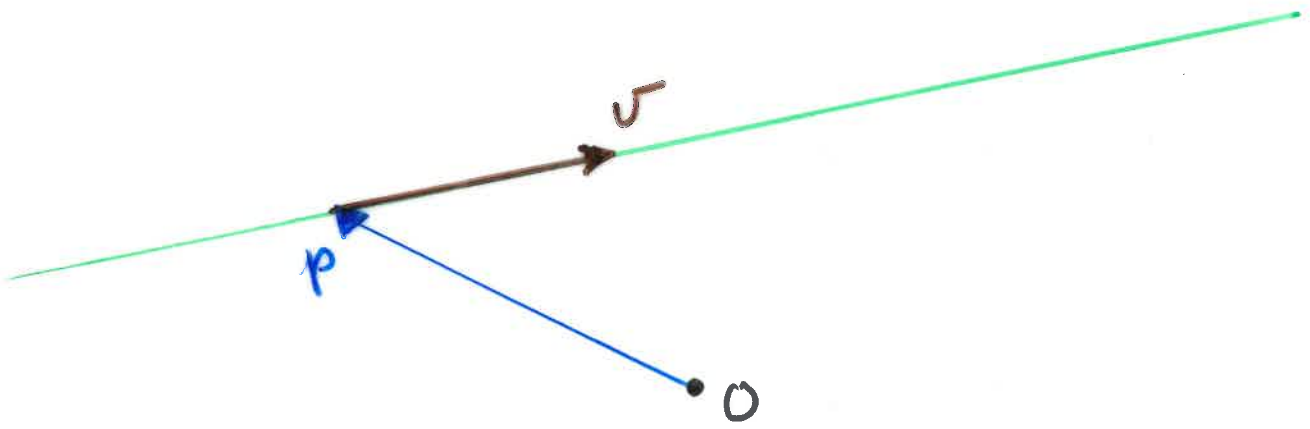


As z changes you move along this line

THE PARAMETRIZATION OF PLANES

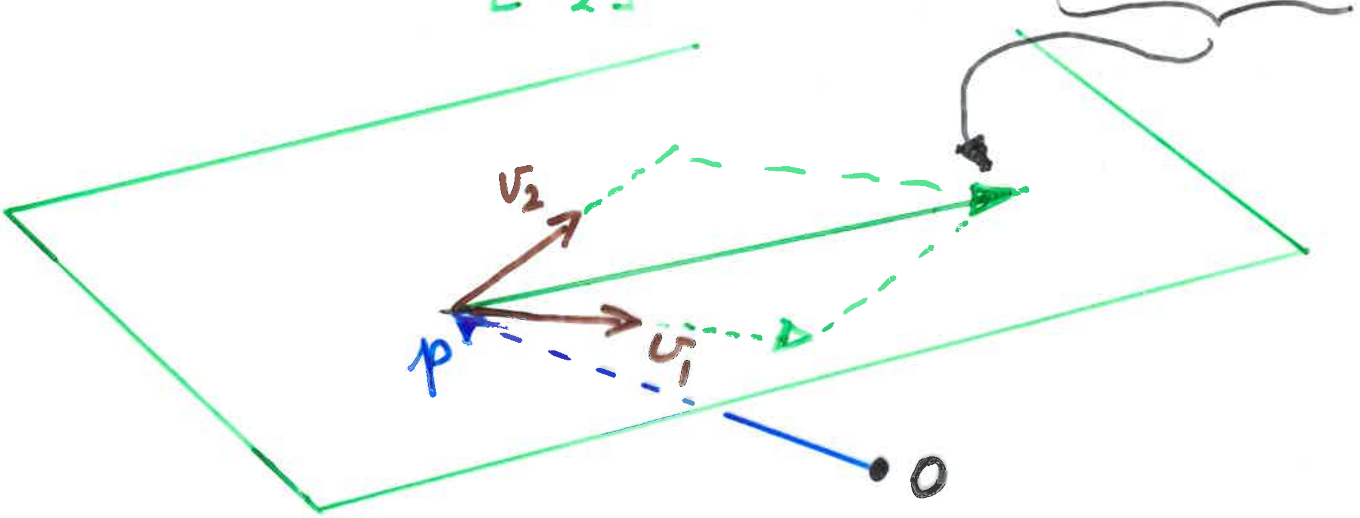
Just as a line through p in the direction v is parametrized by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 : t \longmapsto \gamma(t) = p + t v$$



so also is a plane (in $\mathbb{R}^3$) through p in the directions of $v_1, v_2 \in \mathbb{R}^3$ parametrized by a map

$$\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \longmapsto \gamma(t_1, t_2) = p + t_1 v_1 + t_2 v_2$$



EXAMPLE: Parametrize the plane in $\mathbb{R}^3$ which is determined by the equation

$$1x + 4y - 5z = 2.$$

SOLUTION:

$$\boxed{1x + 4y - 5z = 2}$$

$$\iff \boxed{x = 2 - 4y + 5z}$$

$$\iff \boxed{\begin{aligned} x &= 2 - 4y + 5z \\ y &= 0 + 1y + 0z \\ z &= 0 + 0y + 1z \\ &\text{FOR ALL } y, z \in \mathbb{R} \end{aligned}}$$

The silly equations

$$\Longleftrightarrow \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 0\\ 0\end{bmatrix} + y\begin{bmatrix}-4\\ 1\\ 0\end{bmatrix} + z\begin{bmatrix}5\\ 0\\ 1\end{bmatrix}$$

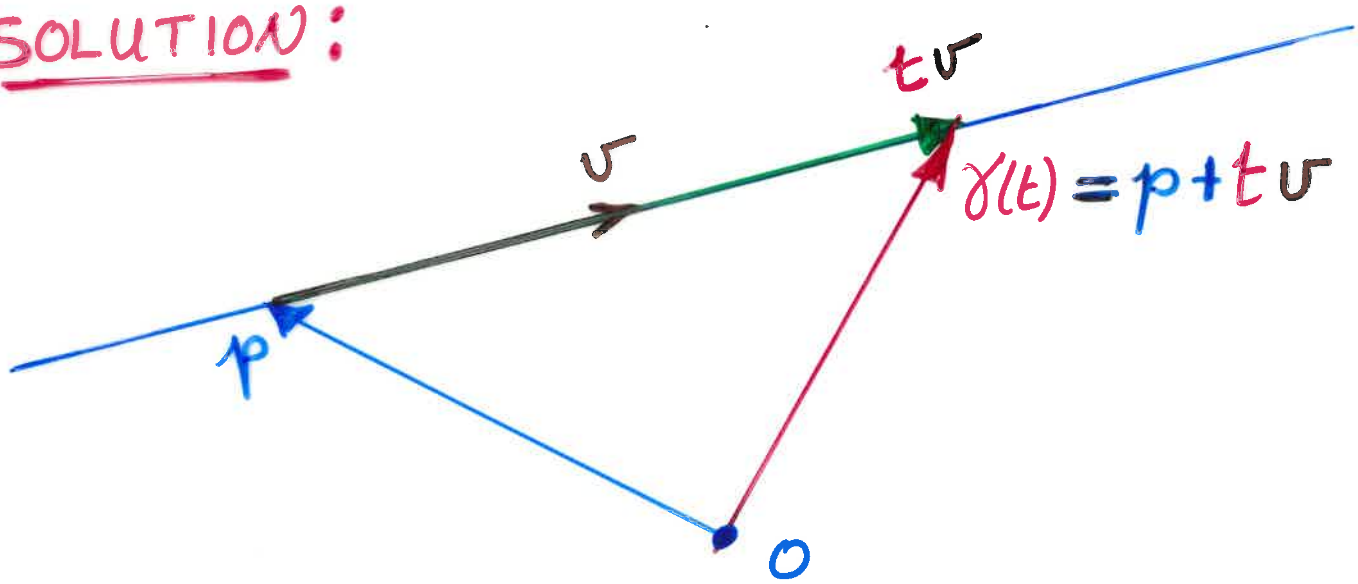
For all $y, z \in \mathbb{R}$

That is, the plane is parametrized by

$$\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : \begin{bmatrix} y\\ z\end{bmatrix} \longmapsto \gamma(y,z) = \underbrace{\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}}_{= p} + y\underbrace{\begin{bmatrix}-4\\ 1\\ 0\end{bmatrix}}_{= v_1} + z\underbrace{\begin{bmatrix}5\\ 0\\ 1\end{bmatrix}}_{= v_2}$$

EXAMPLE 1: Parametrize the line in $\mathbb{R}^3$ which passes through the point $p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the direction $v = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$

SOLUTION:



a parametrization is given by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 : t \longmapsto \gamma(t) = p + tv$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2t \\ 2 - t \\ 3 - 3t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

EXAMPLE 2: Parametrize the line in $\mathbb{R}^3$ which passes through the points

$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

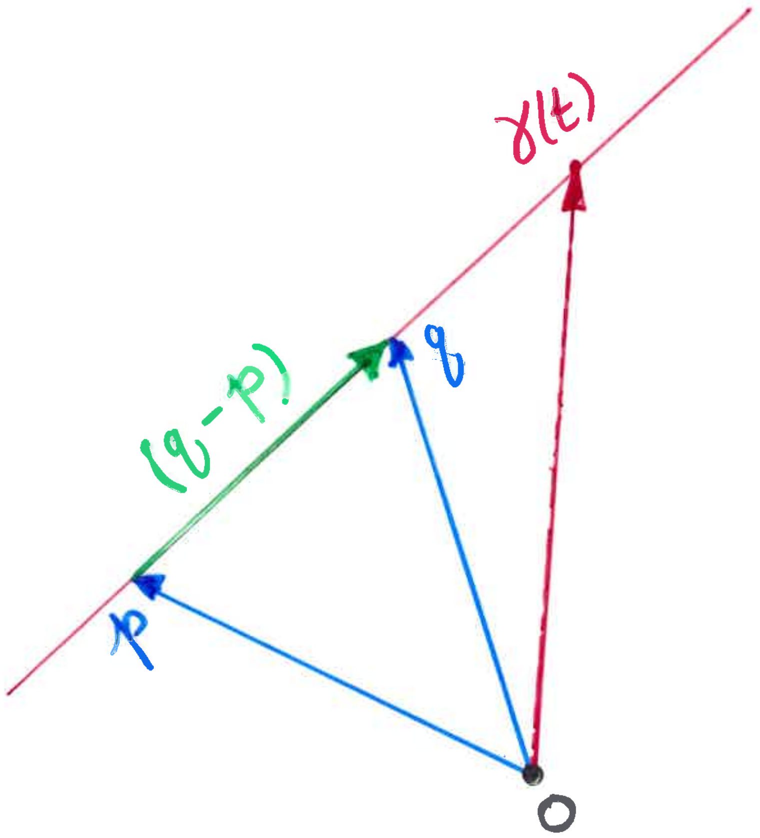
SOLUTION: A parametrization is given by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 : t \longmapsto \gamma(t) = p + t(q-p)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\left(\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$$

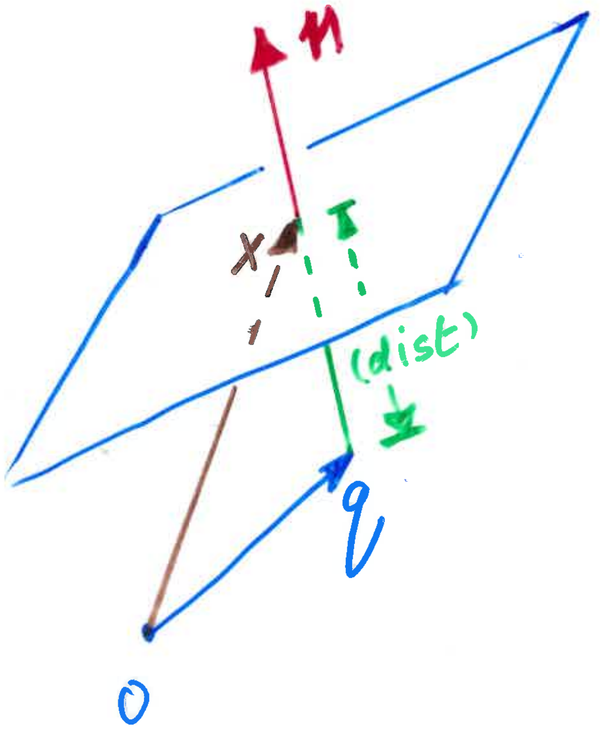
$$= \begin{bmatrix} 1+3t \\ 2-3t \\ 3-t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

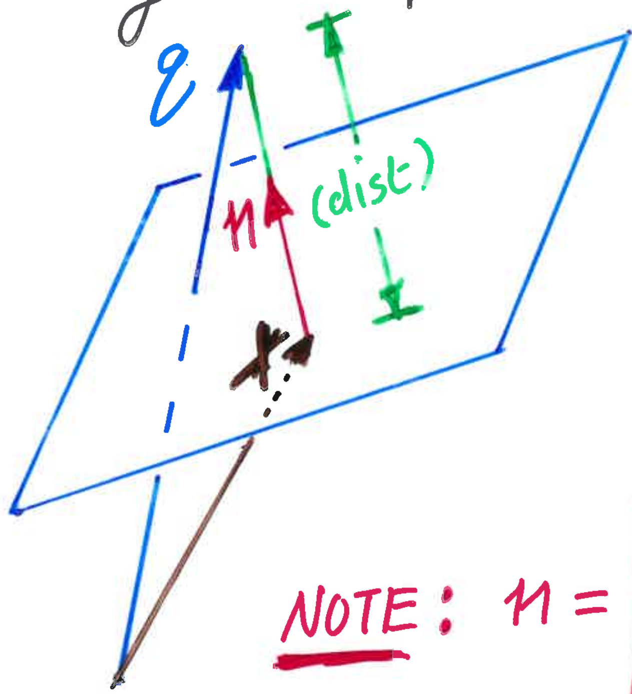


THE PERPENDICULAR DISTANCE FROM A POINT TO A PLANE

Let (dist) = $\left\{ \begin{array}{l} \text{The PERPENDICULAR DISTANCE} \\ \text{from the point } q = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \\ \text{to the plane} \\ ax + by + cz = d \end{array} \right.$

One of the following two pictures hold:





NOTE: $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Here $X = q + (\text{dist})\dfrac{n}{\|n\|}$ is on the plane	Here $X = q + (\text{dist})\left(-\dfrac{n}{\|n\|}\right)$ is on the plane

In either case,

$$X = q + \lambda \frac{n}{\|n\|} \quad \text{where } (\text{dist}) = |\lambda|$$

is on the plane

$$ax + by + cz = d$$

That is,

$$\langle n, X \rangle = d$$

$$\Longrightarrow \left\langle n, q + \lambda \frac{n}{\|n\|} \right\rangle = d$$

$$\Longrightarrow \langle n, q \rangle + \lambda \|n\| = d$$

$$\Longrightarrow (\text{dist}) = |\lambda| = \frac{|\langle n, q \rangle - d|}{\|n\|}$$

This formula should look familiar

$$= \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

In particular

The distance from the ORIGIN to the plane $ax+by+cz=d$ $= \dfrac{|d|}{\|n\|}$

since in this case $x_1 = y_1 = z_1 = 0$

EXAMPLE: Find the distance from the point $q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to the plane $2x-4y+3z=5$.

SOLUTION: with $n = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ we have

$$\text{distance} = \frac{|\langle n, q\rangle - 5|}{\|n\|}$$

$$= \frac{|(2)1+(-4)2+(3)3-5|}{\sqrt{2^2+(-4)^2+(3)^2}}$$

$$= \frac{|-2|}{\sqrt{29}} = \frac{2}{\sqrt{29}} \text{ (units)}$$