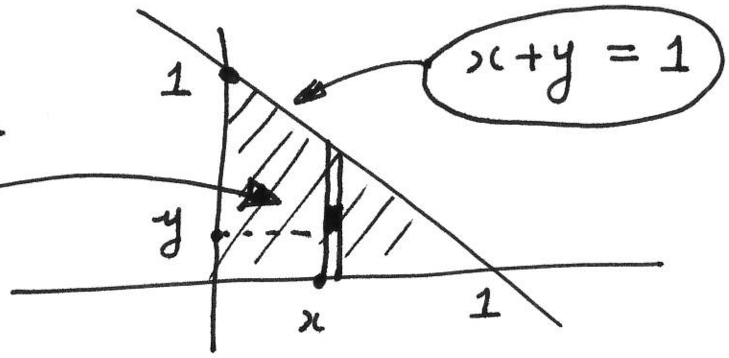
MS221 HOMEWORK SET 10

Q1 First note that the integral

$$\int_0^1 \int_{y=0}^{y=(1-x)} f(x,y)\,dy\,dx \quad \text{corresponds to}$$

an integral over the region Ω:



We are given a change of coordinates

$$\boxed{\begin{aligned} u &= x+y \\ v &= \frac{y}{x+y} \end{aligned}}$$

which we invert to get

$$\boxed{\begin{aligned} x &= u - uv \\ y &= uv \end{aligned}}$$

Under the transformation (i.e. the map) $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ the region Ω is mapped to the region $\tilde{\Omega}$ in the uv-plane which we determine as follows:

The boundary curves of Ω are given by : $x+y=1$, $x=0$ and $y=0$.
The corresponding boundary curves of $\tilde{\Omega}$ are determined according to :

$$\Omega \qquad\qquad \tilde{\Omega}$$

$$\boxed{x+y=1} \longleftrightarrow \boxed{u=1} \quad \left\{ \begin{array}{l} \text{since} \\ u = x+y \end{array} \right.$$

$$\boxed{x=0} \longleftrightarrow \boxed{v=1} \quad \left\{ \begin{array}{l} \text{since} \\ v = \dfrac{y}{x+y} \end{array} \right.$$

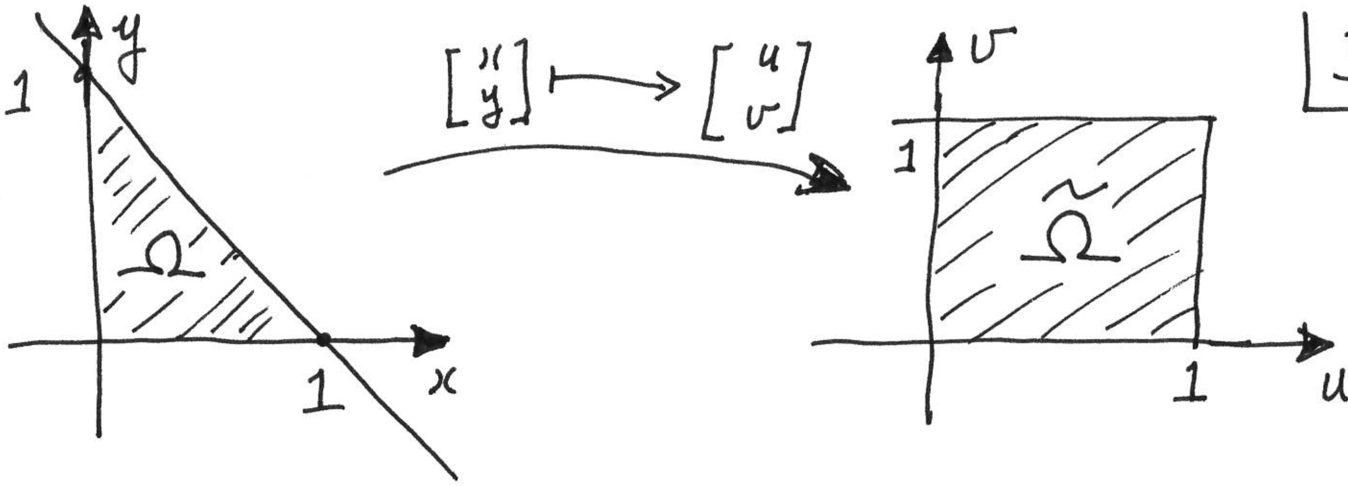
$$\boxed{y=0} \longleftrightarrow \boxed{v=0} \quad \left\{ \begin{array}{l} \text{since} \\ v = \dfrac{y}{x+y} \end{array} \right.$$

$$\boxed{\begin{array}{l} \underline{\text{Note}}\text{: The} \\ \text{origin} \\ (x,y) = (0,0) \end{array}} \longleftrightarrow \boxed{u=0} \quad \left\{ \begin{array}{l} \text{since} \\ u = x+y \end{array} \right.$$

It is important here, if we want to use the given change of coordinates

$$: \Omega \longrightarrow \tilde{\Omega} : \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} u \\ v \end{bmatrix},$$

that the region Ω is given by
$0 < x$, $0 < y$ and $x+y \leq 1$



By the change of variable formula for integration we have that

$$\iint_{\Omega} e^{y/(x+y)}\,dy\,dx = \iint_{\tilde{\Omega}} e^{v} \cdot \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Note:

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$x = u(1-v)$
$y = uv$

$$= \det \begin{bmatrix} (1-v) & -u \\ v & u \end{bmatrix}$$

$$= u - uv + uv$$

$$= u$$

$$= \int_0^1 \int_0^1 e^{v} u \, du\, dv$$

Thus

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx = \int_0^1 e^v \left[\frac{u^2}{2} \right]_{u=0}^{u=1} dv$$

$$= \frac{1}{2} \int_0^1 e^v \, dv$$

$$= \frac{e^v}{2} \Bigg|_{v=0}^{v=1}$$

$$= \frac{e-1}{2} .$$

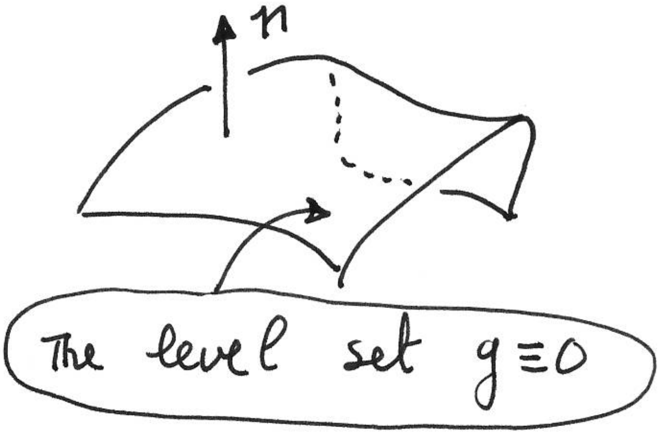
Q2 The surface $\mathcal{S}$ in $\mathbb{R}^3$ is given as the graph

$$z = (x-y)^2 \quad \forall \quad (x,y) \in \Omega .$$

We present this as the level set

$$g(x,y,z) = 0 \quad \text{where} \quad g(x,y,z) = z - (x-y)^2 .$$

The vector field $n = \frac{\nabla g}{\|\nabla g\|}$ is the "upward pointing" unit normal field to $\mathcal{S}$.



Note that

$$\nabla g = \begin{bmatrix} -2(x-y) \\ +2(x-y) \\ 1 \end{bmatrix} .$$

Now,

$$\iint_{\mathcal{S}} \langle F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} \left[\left\langle F, \frac{\nabla g}{\cancel{\|\nabla g\|}} \right\rangle \cancel{\|\nabla g\|} \right]_{z=(x-y)^2} dx\,dy$$

$$= \iint_{\Omega} \left\langle \begin{bmatrix} x+y \\ 0 \\ 2z \end{bmatrix}, \begin{bmatrix} -2(x-y) \\ 2(x-y) \\ 1 \end{bmatrix} \right\rangle_{z=(x-y)^2} dx\,dy$$

Thus

$$\iint_{\mathcal{S}} \langle F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} \Big[-(x+y)2(x-y) + 2z \Big]_{z=(x-y)^2} \, dx\,dy$$

$$= \iint_{\Omega} 2(x-y)\Big[-(x+y) + (x-y) \Big] \, dx\,dy$$

$$= \iint_{\Omega} 4y(y-x) \, dx\,dy \, .$$

So the required function f is :

$$f : \Omega \longrightarrow \mathbb{R} : (x,y) \longmapsto f(x,y) = 4y(y-x).$$

Q3 Since the domain of F is $\mathbb{R}^3$ which is simply-connected;

$$\boxed{F \text{ is conservative}} \iff \boxed{\nabla \times F = 0}.$$

Here

$$\nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & (x\cos y + \sin z) & y \cos z \end{vmatrix}$$

$$= \begin{bmatrix} \cos z - \cos z \\ 0 - 0 \\ \cos y - \cos y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that F is conservative. To find the scalar potential $\phi : \mathbb{R}^3 \to \mathbb{R}$ we must solve

$$\nabla \phi = F$$

for the function ϕ. That is, we

must solve

$$\frac{\partial \phi}{\partial x}(x,y,z) = \sin y \quad \ldots\ldots\ldots \text{(A)}$$

$$\frac{\partial \phi}{\partial y} = x\cos y + \sin z \quad \ldots\ldots \text{(B)}$$

$$\frac{\partial \phi}{\partial z} = y\cos z \quad \ldots\ldots \text{(C)}$$

$$\overset{\text{(A)}}{\Longrightarrow} \phi(x,y,z) = x\sin y + \psi(y,z) \quad \ldots \text{(D)}$$

by (B)

$$\cancel{x\cos y} + \sin z = \frac{\partial \phi}{\partial y} = \cancel{x\cos y} + \frac{\partial \psi}{\partial y}(y,z)$$

Thus $\dfrac{\partial \psi(y,z)}{\partial y} = \sin z$

so that $\psi(y,z) = y\sin z + \chi(z)$

$$\overset{\text{(D)}}{\Longrightarrow} \phi(x,y,z) = x\sin y + y\sin z + \chi(z) \ldots \text{(E)}$$

We proceed as we did in the previous step:

$$\phi(x,y,z) = x\sin y + y\sin z + \chi(z)$$

$$\underbrace{\cancel{y\cos z}}_{\text{by (C)}} = \frac{\partial\phi}{\partial z} = 0 + \cancel{y\cos z} + \frac{d}{dz}\chi(z)$$

Thus $\dfrac{d}{dz}\chi(z) = 0$

so that $\chi(z) = C$ a constant

Finally

$$\overset{(E)}{\Longrightarrow} \phi(x,y,z) = x\sin y + y\sin z + C.$$