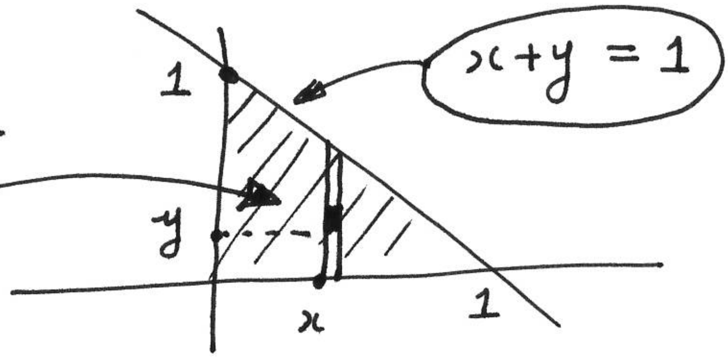
MS221 HOMEWORK SET 10

Q1 First note that the integral

$$\int_0^1 \int_{y=0}^{y=(1-x)} f(x,y)\,dy\,dx$$ corresponds to

an integral over the region Ω:



We are given a change of coordinates

$$\boxed{\begin{aligned} u &= x+y \\ v &= \frac{y}{x+y} \end{aligned}}$$

which we invert to get

$$\boxed{\begin{aligned} x &= u-uv \\ y &= uv \end{aligned}}$$

Under the transformation (i.e. the map)

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$ the region Ω is

mapped to the region $\tilde{\Omega}$ in the uv-plane which we determine as follows:

The boundary curves of Ω are given by: $x+y=1$, $x=0$ and $y=0$. The corresponding boundary curves of $\tilde{\Omega}$ are determined according to:

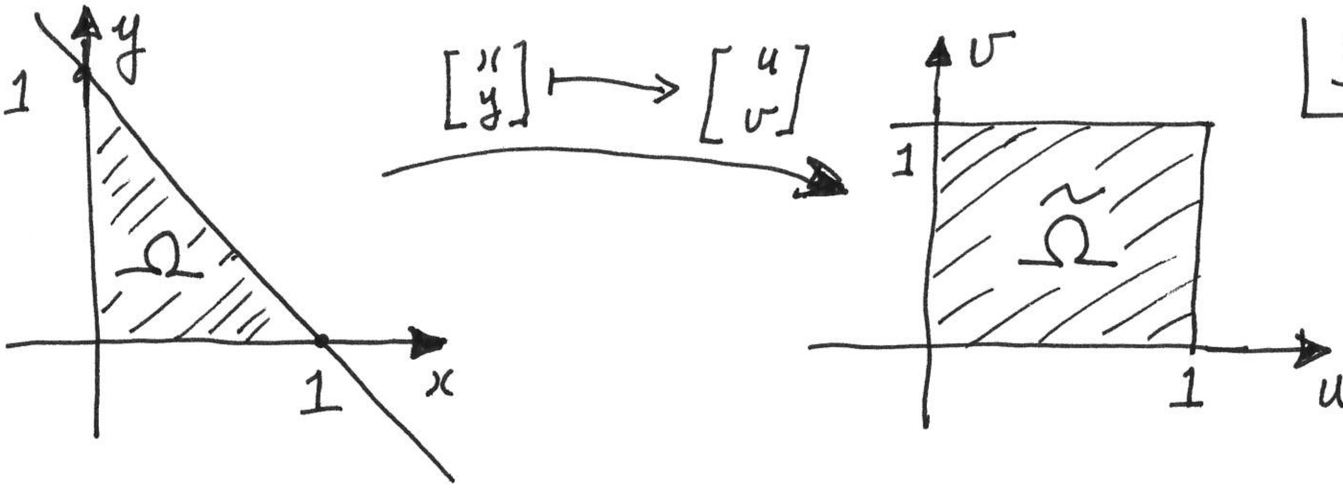
Ω		$\tilde{\Omega}$	
$\boxed{x+y=1}$	$\longleftrightarrow$	$\boxed{u=1}$	since $u = x+y$
$\boxed{x=0}$	$\longleftrightarrow$	$\boxed{v=1}$	since $v = \dfrac{y}{x+y}$
$\boxed{y=0}$	$\longleftrightarrow$	$\boxed{v=0}$	since $v = \dfrac{y}{x+y}$
Note: The origin $(x,y)=(0,0)$	$\longleftrightarrow$	$\boxed{u=0}$	since $u = x+y$

It is important here, if we want to use the given change of coordinates

$$: \Omega \longrightarrow \tilde{\Omega} : \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} u \\ v \end{bmatrix},$$

that the region Ω is given by

$$0 < x, \quad 0 < y \quad \text{and} \quad x+y \leq 1$$



By the change of variable formula for integration we have that

$$\iint_{\Omega} e^{y/(x+y)}\, dy\, dx = \iint_{\tilde{\Omega}} e^{v} \cdot \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Note:

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$x = u(1-v)$
$y = uv$

$$= \det \begin{bmatrix} (1-v) & -u \\ v & u \end{bmatrix}$$

$$= u - uv + uv$$

$$= u$$

$$= \int_0^1 \int_0^1 e^{v} u \, du\, dv$$

Thus

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \int_0^1 e^v \left[\frac{u^2}{2}\right]_{u=0}^{u=1} dv$$

$$= \frac{1}{2}\int_0^1 e^v\,dv$$

$$= \frac{e^v}{2}\Bigg|_{v=0}^{v=1}$$

$$= \frac{e-1}{2}.$$

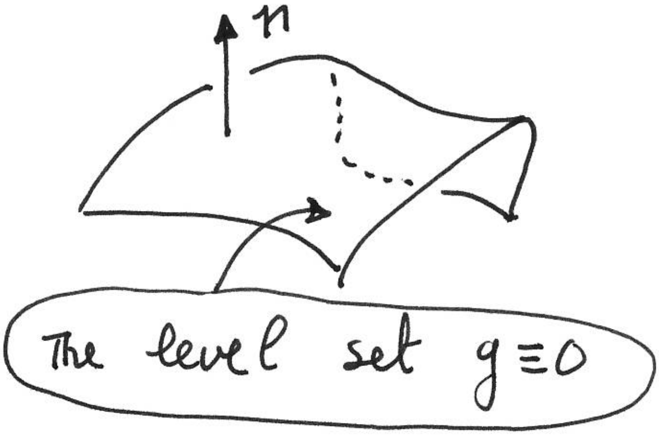
Q2 The surface $\mathcal{S}$ in $\mathbb{R}^3$ is given as the graph

$$z = (x-y)^2 \quad \forall \; (x,y) \in \Omega .$$

We present this as the level set

$$g(x,y,z) = 0 \quad \text{where } g(x,y,z) = z - (x-y)^2 .$$

The vector field $n = \dfrac{\nabla g}{\|\nabla g\|}$ is the "upward pointing" unit normal field to $\mathcal{S}$.



Note that

$$\nabla g = \begin{bmatrix} -2(x-y) \\ +2(x-y) \\ 1 \end{bmatrix} .$$

Now,

$$\iint_{\mathcal{S}} \langle F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} \left[\left\langle F, \frac{\nabla g}{\cancel{\|\nabla g\|}} \right\rangle \cancel{\|\nabla g\|} \right]_{z=(x-y)^2} dx dy$$

$$= \iint_{\Omega} \left\langle \begin{bmatrix} x+y \\ 0 \\ 2z \end{bmatrix}, \begin{bmatrix} -2(x-y) \\ 2(x-y) \\ 1 \end{bmatrix} \right\rangle_{z=(x-y)^2} dx dy$$

Thus

$$\iint_{\mathcal{S}} \langle F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} \Big[-(x+y)2(x-y) + 2z \Big]_{z=(x-y)^2} dx\,dy$$

$$= \iint_{\Omega} 2(x-y)\Big[-(x+y) + (x-y)\Big] dx\,dy$$

$$= \iint_{\Omega} 4y(y-x)\, dx\,dy \, .$$

So the required function f is :

$$f : \Omega \longrightarrow \mathbb{R} : (x,y) \longmapsto f(x,y) = 4y(y-x).$$

Q3 Since the domain of F is $\mathbb{R}^3$ which is simply-connected;

$$\boxed{F \text{ is conservative}} \iff \boxed{\nabla \times F = 0}.$$

Here

$$\nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & (x\cos y + \sin z) & y\cos z \end{vmatrix}$$

$$= \begin{bmatrix} \cos z - \cos z \\ 0 - 0 \\ \cos y - \cos y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that F is conservative. To find the scalar potential $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ we must solve

$$\nabla \phi = F$$

for the function ϕ. That is, we

must solve

$$\frac{\partial\phi}{\partial x}(x,y,z) = \sin y \quad \ldots\ldots\ldots\ldots (A)$$

$$\frac{\partial\phi}{\partial y} = x\cos y + \sin z \quad \ldots\ldots\ldots (B)$$

$$\frac{\partial\phi}{\partial z} = y\cos z \quad \ldots\ldots\ldots\ldots (C)$$

$$\overset{(A)}{\Longrightarrow} \phi(x,y,z) = x\sin y + \psi(y,z) \ldots\ldots (D)$$

$$\cancel{x\cos y} + \sin z \overset{\text{by (B)}}{=} \frac{\partial\phi}{\partial y} = \cancel{x\cos y} + \frac{\partial\psi}{\partial y}(y,z)$$

Thus $\dfrac{\partial\psi(y,z)}{\partial y} = \sin z$

so that $\psi(y,z) = y\sin z + \chi(z)$

$$\overset{(D)}{\Longrightarrow} \phi(x,y,z) = x\sin y + y\sin z + \chi(z) \ldots\ldots (E)$$

We proceed as we did in the previous step:

$$\phi(x,y,z) = x\sin y + y\sin z + \chi(z)$$

$$\underbrace{\cancel{y\cos z}}_{\text{by (C)}} = \frac{\partial \phi}{\partial z} = 0 + \cancel{y\cos z} + \frac{d}{dz}\chi(z)$$

Thus $\frac{d}{dz}\chi(z) = 0$

so that $\chi(z) = C$ a constant

Finally

$$\overset{(E)}{\Longrightarrow} \phi(x,y,z) = x\sin y + y\sin z + C.$$

Q4 $\mathcal{S}$ is the surface in $\mathbb{R}^3$ which is given by

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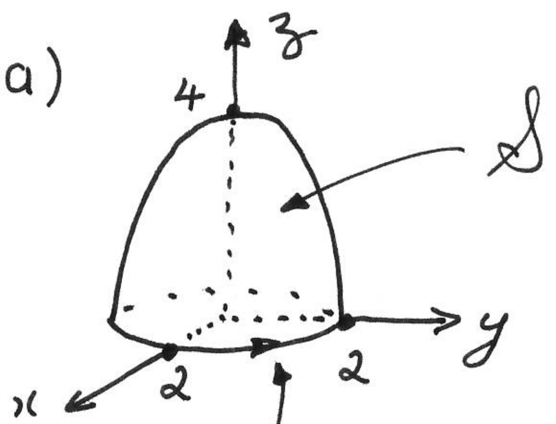
$$z = 4 - (x^2+y^2) \quad \forall\ (x,y) \text{ s.t. } x^2+y^2 \leq 4$$

(a)



The boundary to $\mathcal{S}$ is the curve
$$\mathcal{C} = \begin{cases} x^2+y^2 = 4 \\ z = 0 \end{cases}$$

The easiest way to see that $\mathcal{S}$ is as show is to observe that

(i) $x = y = 0 \Rightarrow z = 4$

(ii) If we examine the horizontal sections $z = z_0$ (say) we get circles $\begin{cases} x^2+y^2 = 4 - z_0 \\ z = z_0 \end{cases}$.

(b) View the surface $\mathcal{S}$ as the level set

$$g(x,y,z) = 0 \quad \text{where } g(x,y,z) = z + x^2 + y^2 - 4.$$

$\Rightarrow n = \frac{1}{\|\nabla g\|} \cdot \nabla g$ is the unit upward-pointing normal field along $\mathcal{S}$

Here $\nabla g = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \Rightarrow \|\nabla g\| = \sqrt{1+4x^2+4y^2}$

$$\Rightarrow n = \frac{1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}.$$

(c) $$F = \begin{bmatrix} x + yz \\ y + xz \\ xyz \end{bmatrix}$$

$$\Rightarrow \nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+yz) & (y+xz) & xyz \end{vmatrix}$$

$$= \begin{bmatrix} xz - x \\ y - yz \\ z - z \end{bmatrix} = \begin{bmatrix} xz - x \\ y - yz \\ 0 \end{bmatrix}.$$

Thus

$$\iint_{\mathcal{S}} \langle \nabla \times F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} \left\langle \nabla \times F, \frac{\nabla g}{\cancel{\|\nabla g\|}} \right\rangle \cancel{\|\nabla g\|} \, dx\,dy \Big|_{z = 4 - (x^2+y^2)}$$

$$= \iint_{\Omega} \left\langle \begin{bmatrix} xz - x \\ y - yz \\ 0 \end{bmatrix}, \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \right\rangle dx\,dy \Big|_{z = 4 - (x^2+y^2)}$$

$$= \iint_{\Omega} \left[2x^2 z - 2x^2 + 2y^2 - 2y^2 z \right] dx\,dy \Big|_{z = 4 - (x^2+y^2)}$$

$$\Rightarrow \iint_{\mathcal{S}} \langle \nabla \times F, n \rangle \, dA_{\mathcal{S}} = \iint_{\Omega} 2(x^2 - y^2)[z-1] \, dx dy$$

$z = 4 - (x^2+y^2)$

$$= \iint_{\Omega} \underbrace{2(x^2-y^2)\left[3-(x^2+y^2)\right]}_{\varphi(x,y)} dx dy \; .$$

(d) By Stokes' Theorem

$$\iint_{\mathcal{S}} \langle \nabla \times F, n \rangle \, dA_{\mathcal{S}} = \int_{\mathcal{C}} \langle F, \tau \rangle \, ds$$

where $\mathcal{C}$ is the boundary of $\mathcal{S}$. That is $\mathcal{C}$ is the circle $x^2+y^2=4$ on the xy-plane

$$= \int_0^{2\pi} \left\langle F(\gamma(t)), \frac{d\gamma}{dt} \right\rangle dt$$

where

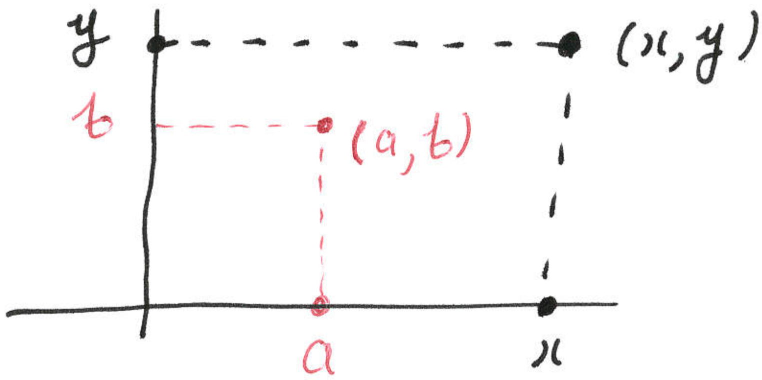
$$\gamma : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto \begin{bmatrix} 2\cos t \\ 2\sin t \\ 0 \end{bmatrix}$$

$$= \int_0^{2\pi} \left\langle \begin{bmatrix} 2\cos t \\ 2\sin t \\ 0 \end{bmatrix}, \begin{bmatrix} -2\sin t \\ 2\cos t \\ 0 \end{bmatrix} \right\rangle dt$$

$$\Rightarrow \iint_{\mathcal{S}} \langle \nabla \times F, n \rangle \, dA_{\mathcal{S}} = \int_0^{2\pi} \underbrace{\left[-4\cos t \sin t + 4 \sin t \cos t \right]}_{\substack{\parallel \\ 0}} dt$$

$$= 0 .$$

Q5



Taylor's Theorem allows us to use alot of information that we have about f at the point (a, b) to <u>PREDICT</u> the value of f at the point (x, y) which (usually) we think of as being <u>near</u> (a, b). So (usually) we have in mind that

BOTH $(x - a)$ AND $(y - b)$

are small. The prediction is given by :

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$
$$+ \frac{1}{2}\left\{ \frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2 \right\}$$

+ higher order terms in $(x-a)$ & $(y-b)$.

We apply this to the function

$$f : \mathbb{R}^2 \to \mathbb{R} : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto f(x,y) = x^3 - y^2 + y .$$

NOTE: In this case $f(x,y)$ is given by a very simple formula which we can calculate easily at any (x,y) so our use of Taylor's Theorem up to second order terms is for no more than the purpose of illustration.

Proceed as follows:

At (x,y)	At $(a,b) = (2,-3)$
$f(x,y) = x^3 - y^2 + y$	$f(2,-3) = 8-9-3 = -4$
$\frac{\partial f}{\partial x}(x,y) = 3x^2$	$\frac{\partial f}{\partial x}(2,-3) = 12$
$\frac{\partial f}{\partial y}(x,y) = -2y+1$	$\frac{\partial f}{\partial y}(2,-3) = 7$
$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$	$\frac{\partial^2 f}{\partial x^2}(2,-3) = 12$
$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$	$\frac{\partial^2 f}{\partial x \partial y}(2,-3) = 0$
$\frac{\partial^2 f}{\partial y^2}(x,y) = -2$	$\frac{\partial^2 f}{\partial y^2}(2,-3) = -2$

Thus

$$x^3 - y^2 + y = -4 + 12(x-2) + 7(y+3)$$
$$+ \frac{1}{2}\left\{ 12(x-2)^2 - 2(y+3)^2 \right\}$$

+ higher order terms in $(x-2)$ and $(y+3)$.

Q6

We proceed as in Q5:

At (x, y)	At $(a, b) = (0, -1)$
$f(x,y) = \sin(xy)$	$f(0,-1) = 0$
$\frac{\partial f}{\partial x}(x,y) = y\cos(xy)$	$\frac{\partial f}{\partial x}(0,-1) = -1$
$\frac{\partial f}{\partial y}(x,y) = x\cos(xy)$	$\frac{\partial f}{\partial y}(0,-1) = 0$
$\frac{\partial^2 f}{\partial x^2}(x,y) = -y^2\sin(xy)$	$\frac{\partial^2 f}{\partial x^2}(0,-1) = 0$
$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \cos(xy) - xy\sin(xy)$	$\frac{\partial^2 f}{\partial x \partial y}(0,-1) = 1$
$\frac{\partial^2 f}{\partial y^2}(x,y) = -x^2\sin(xy)$	$\frac{\partial^2 f}{\partial y^2}(0,-1) = 0$

Thus

$$\sin(xy) = -1\,x + \frac{1}{2}\left\{ 2(1)\,x\,(y+1) \right\}$$

+ higher order terms in x and $(y+1)$.