MS 221 — Homework Set (2)

QUESTION 1

In the case of the vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \in \boldsymbol{R}^3$ calculate $\|\boldsymbol{v}_1\|$, $\|\boldsymbol{v}_2\|$, $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$ and hence determine the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

QUESTION 2

In the case of the vectors $\boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in \boldsymbol{R}^3$ find the constant $\alpha \in \boldsymbol{R}$ and the vector $\boldsymbol{x}^{\perp} \in \boldsymbol{R}^3$ which is **perpendicular** to the (unit) vector $\boldsymbol{u}$ such that

$$\boldsymbol{x} = \alpha\, \boldsymbol{u} + \boldsymbol{x}^{\perp}$$

QUESTION 3

Consider the following vectors in $\boldsymbol{R}^3$:

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Now show that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ are **orthonormal** (with respect to the usual innerproduct in $\boldsymbol{R}^3$) and hence express the vector $\boldsymbol{v}$ as a linear combination of them.

QUESTION 4

Find an equation of the plane in $\boldsymbol{R}^3$ which passes through the point $\boldsymbol{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and is perpendicular to the vector $\boldsymbol{n} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

QUESTION 5

Find the perpendicular distance from the plane $2x - y + z = 12$ to the origin.

QUESTION 6

Find the cross product of the vectors in $\boldsymbol{R}^3$

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

and hence determine the equation of the plane in $\boldsymbol{R}^3$ which passes through the origin, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

QUESTION 7

Find the equation of the plane in $\boldsymbol{R}^3$ which passes through the points

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

QUESTION 8

Find the area of the ~~rectangle~~ Parallelogram which is spanned by the vectors

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \boldsymbol{R}^3$$

QUESTION 9

Let $\mathcal{P}$ be the plane through the origin which is perpendicular to the (unit) vector $\boldsymbol{\xi} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. When light from infinity, shining parallel to $\boldsymbol{\xi}$, falls on the ~~rectangle~~ Parallelogram described in question 8 it casts a shadow on the plane $\mathcal{P}$. Find the area of this shadow.

MS 221 HOMEWORK SET 2

Q1

$$\|v_1\| = \sqrt{1+1+4} = \sqrt{6}$$

$$\|v_2\| = \sqrt{1+9+1} = \sqrt{11}$$

$$\langle v_1, v_2 \rangle = -1 - 3 + 2 = -2$$

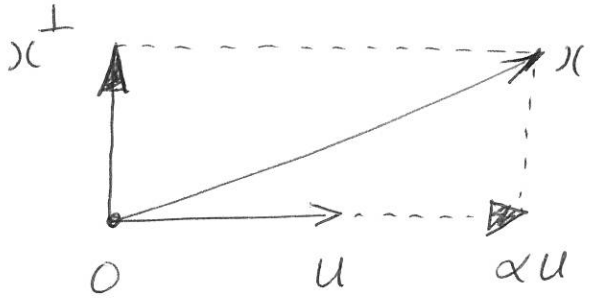
The angle θ between v_1 & v_2 is given by

$$\theta = \cos^{-1}\left(\frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|}\right)$$

$$= \cos^{-1}\left(-\frac{2}{\sqrt{66}}\right)$$

$$= 104^{\circ}\ 15'$$

$\boxed{Q2}$



$$x = \alpha u + x^\perp$$

$$\Rightarrow \langle x, u \rangle = \alpha \underbrace{\langle u, u \rangle}_{= 1} + \underbrace{\langle x^\perp, u \rangle}_{= 0}$$

$$\Rightarrow \alpha = \langle x, u \rangle = \left\langle \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle$$

$$= \frac{1}{\sqrt{2}} (2 - 3) \quad = -\frac{1}{\sqrt{2}}$$

$$\boxed{\alpha = -1/\sqrt{2}}$$

$$x^\perp = x - \alpha u \quad = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \left(-\frac{1}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 4 + 1 \\ 6 - 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}$$

$\boxed{Q3}$ To show that $u_1\ u_2\ u_3$ are ORTHONORMAL just check that

$$\langle u_i , u_j \rangle = \begin{cases} 0 & \forall\ i \neq j \\ 1 & \forall\ i = j \end{cases}$$

Since $u_1\ u_2\ u_3$ are orthonormal vectors, we have $\forall\ v \in \mathbb{R}^3$ that

$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle v, u_3 \rangle u_3 .$$

In this case,

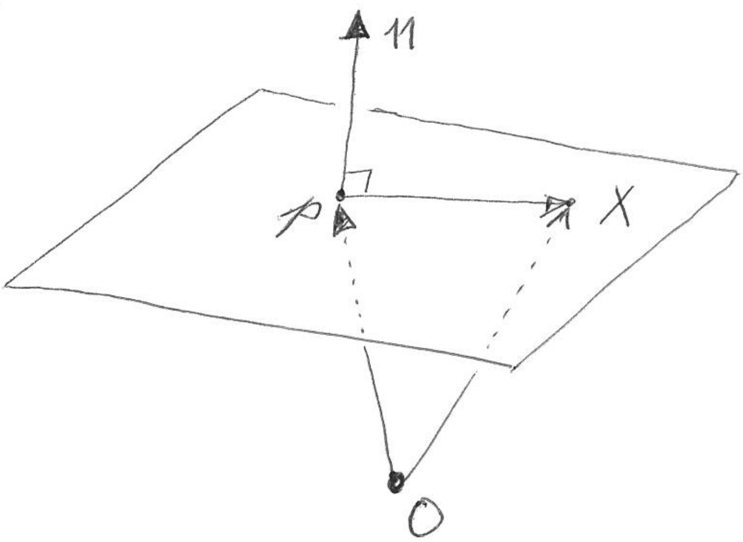
$$\langle v, u_1 \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}} (1+3) = 2\sqrt{2}$$

$$\langle v, u_2 \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{3}} (-1-2+3) = 0$$

$$\langle v, u_3 \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{6}} (1-4-3) = -\sqrt{6}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2\sqrt{2}\, u_1 - \sqrt{6}\, u_3$$

$\boxed{Q4}$



$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X \in \text{Plane} \iff \langle X - p, n \rangle = 0$$

$$\iff \left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 0$$

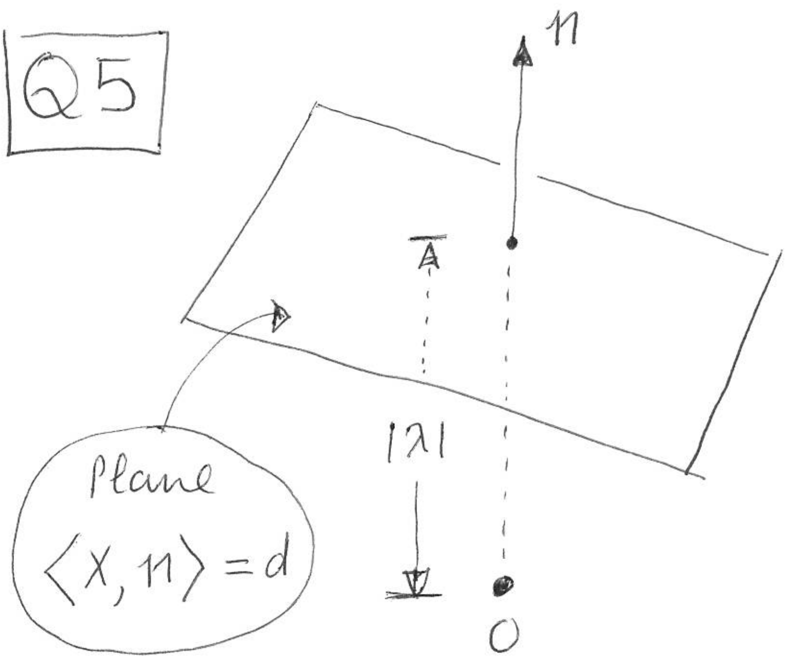
$$\iff \left\langle \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 0$$

$$\iff -3(x-1) + 2(y-2) + (z-3) = 0$$

$$\iff -3x + 2y + z + 3 - 4 - 3 = 0$$

$$\iff 3x - 2y - z + 4 = 0$$

$\boxed{Q5}$



If $\lambda \in \mathbb{R}$ is such that

$$\lambda \frac{n}{\|n\|} \in \text{Plane}$$

then

$$|\lambda| = \left\{ \begin{array}{l} \text{Perpendicular} \\ \text{distance from} \\ \text{the origin to} \\ \text{the plane.} \end{array} \right.$$

$$\lambda \frac{n}{\|n\|} \in \text{Plane} \iff \left\langle \lambda \frac{n}{\|n\|}, n \right\rangle = d$$

$$\iff \frac{\lambda}{\|n\|} \langle n, n \rangle = d$$

$$\iff \lambda \|n\| = d$$

$$\iff \lambda = \frac{d}{\|n\|}$$

In this case, the plane has equation

$$2x - y + z = 12$$

$n = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $d = 12$. Therefore,

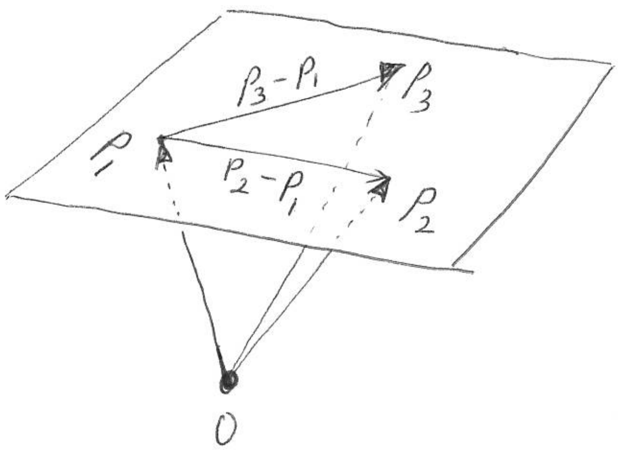
$$\text{distance} = |\lambda| = \frac{|d|}{\|n\|} = \frac{12}{\sqrt{4+1+1}} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

Q6

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 1 & 0 \\ 1 & 3 & -1 \end{vmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

Q7

Put $p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $p_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $p_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



The vector

$$n = (p_2 - p_1) \times (p_3 - p_1)$$

is normal to the plane

Here $n = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{(p_2 - p_1) \times (p_3 - p_1)} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and $\langle X - p_1 , n \rangle = 0$ is an equation of the required plane.

i.e. $\left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 0$

$\Longrightarrow$ $x - 1 = 0$ or $\boxed{x = 1}$

Q8 The area of the parallelogram spanned by x & u is given by

$$\text{area} = \| x \times u \|$$

Here

$$x \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \text{area} = \| x \times u \| = \sqrt{25+4+9} = \sqrt{38}$$

Q9 Put $n = \dfrac{x \times u}{\| x \times u \|}$, then

$$\text{area (of shadow)} = \left(\begin{matrix}\text{area of} \\ \text{Parallelogram}\end{matrix}\right) \left| \langle n, \xi \rangle \right|$$

$$= \| x \times u \| \left| \left\langle \frac{x \times u}{\| x \times u \|}, \xi \right\rangle \right|$$

$$= \langle x \times u, \xi \rangle$$

$$= \left\langle \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$$= \frac{1}{\sqrt{2}} (5 - 3) = \sqrt{2}.$$