MS 221 — Homework Set (3)

QUESTION 1

Parametrize the line joining the points $\boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\boldsymbol{q} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \in \boldsymbol{R}^3$.

QUESTION 2

Parametrize the line in $\boldsymbol{R}^3$ which is determined by the intersection of the planes

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 2 \\ 3x & + & 5y & + & z & = & 1 \end{array}$$

QUESTION 3

Parametrize the circle on the xy-plane which has centre $(3, -5)$ and radius $= 2$.

QUESTION 4

Parametrize the **ellipse** on the xy-plane which is determined by the equation:

$$\frac{(x-1)^2}{5^2} + \frac{(y-3)^2}{7^2} = 1.$$

QUESTION 5

Parametrize the **hyperbola** on the xy-plane which is determined by the equation:

$$\frac{(x-1)^2}{5^2} - \frac{(y-3)^2}{7^2} = 1.$$

QUESTION 6

Parametrize the **parabola** on the xy-plane which is determined by the equation:

$$(y+1)^2 = 12(x-5).$$

QUESTION 7

Find $\lim_{t\to 1} \gamma(t)$ where $\gamma : \boldsymbol{R} \setminus \{1\} \to \boldsymbol{R}^2 : t \mapsto \begin{bmatrix} \dfrac{t^3 - t}{t-1} \\ \\ \dfrac{\sin(t-1)}{t-1} \end{bmatrix}$.

QUESTION 8

If $\boldsymbol{C}$ is the curve in $\boldsymbol{R}^3$ which is parametrized by

$$\gamma : \boldsymbol{R} \to \boldsymbol{R}^3 : t \mapsto \begin{bmatrix} t^3 - t \\ (3t+5)^2 \\ t^2 + 1 \end{bmatrix}$$

do the following:

(a) Calculate $\gamma(-1)$.

(b) Calculate $\dfrac{d\gamma}{dt}(t)$ when $t = -1$.

(c) Parametrize the tangent **LINE** to the curve $\boldsymbol{C}$ at the point $\gamma(-1)$.

QUESTION 9

Let $\gamma(t)$ be as given in Question 8. If the **position** vector of a particle at time t is $\gamma(t)$ find the **velocity** and **acceleration** vectors of this particle at time t.

QUESTION 10

Fix an origin $\mathbf{0}$ (in 3-dimensional space) and let $\boldsymbol{r}(t)$ be the position vector (relative to $\mathbf{0}$) of a particle p at time t. We define:

(a) $\boldsymbol{v}(t) := \dfrac{d\boldsymbol{r}}{dt}(t)$ the velocity vector of p at time t.

(b) $\boldsymbol{M} := m\boldsymbol{v}$ the momentum vector, note ($m =$ mass of p)

(c) $\boldsymbol{F} :=$ the force on p.

(d) $\boldsymbol{A_q} := \boldsymbol{x_q}(t) \times \boldsymbol{M}$ called the angular momentum of the particle p about the fixed point $\boldsymbol{q}$. The vector $\boldsymbol{x_q}(t) := \boldsymbol{r}(t) - \boldsymbol{q}$ is the position vector of the particle p **relative to point** $\boldsymbol{q}$ at time t.

(e) Torque about $\boldsymbol{q} := \boldsymbol{x_q}(t) \times \boldsymbol{F}$.

Newton's Second Law states:

$$\boldsymbol{F} = \frac{d\boldsymbol{M}}{dt}$$

and **The Principle of Angular Momentum** states:

$$\frac{d\boldsymbol{A_q}}{dt} = \boldsymbol{x_q} \times \boldsymbol{F} \quad \text{for every point } \boldsymbol{q} \text{ in 3-space.}$$

Show that these laws are equivalent.

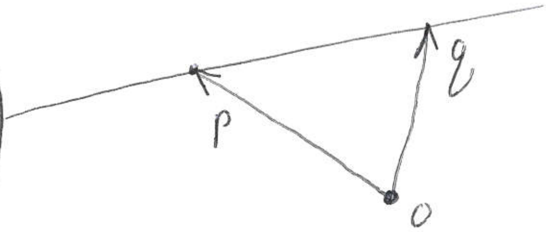
MS 221 HOMEWORK SET (3)

Q1

$$\gamma : \mathbb{R} \to \mathbb{R}^3 : t \mapsto \gamma(t) = p + t(q-p)$$

Thus

$$\gamma(t) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right)$$



$$\Rightarrow \gamma(t) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2t \\ -1+4t \\ 2-t \end{bmatrix}$$

Q2

$$\boxed{\begin{aligned} x + 2y - z &= 2 \\ 3x + 5y + z &= 1 \end{aligned}} \xrightarrow{(Equ^n)_2 - 3(Equ^n)_1} \boxed{\begin{aligned} x + 2y - z &= 2 \\ -y + 4z &= -5 \end{aligned}}$$

$$\downarrow (-1) \times Equ^n_2$$

$$\boxed{\begin{aligned} x \quad + 7z &= -8 \\ y - 4z &= 5 \end{aligned}} \xleftarrow{(Equ^n)_1 - 2(Equ^n)_2} \boxed{\begin{aligned} x + 2y - z &= 2 \\ y - 4z &= 5 \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} x &= -8 - 7z \\ y &= 5 + 4z \\ z &= 0 + 1z \end{aligned}} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ 0 \end{bmatrix} + z \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

z is the parameter

Q3 Circle with centre $(3,-5)$ Radius $= 2$

$$\gamma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 + 2\cos t \\ -5 + 2\sin t \end{bmatrix}$$

Q4 Ellipse $\left(\frac{x-1}{5}\right)^2 + \left(\frac{y-3}{7}\right)^2 = 1$

$$\gamma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 + 5\cos t \\ 3 + 7\sin t \end{bmatrix}$$

Q5 Hyperbola $\left(\frac{x-1}{5}\right)^2 - \left(\frac{y-3}{7}\right)^2 = 1$

$$\gamma : \mathbb{R} \to \mathbb{R}^2 : t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 + 5\cosh t \\ 3 + 7\sinh t \end{bmatrix}$$

Note: This prarametrizes the right hand branch only.

Q6 Parabola $(y+1)^2 = 4(3)(x-5)$

$$\gamma : \mathbb{R} \to \mathbb{R}^2 : t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 5 + 3t^2 \\ -1 + 6t \end{bmatrix}$$

Note: For $Y^2 = 4aX$ put $X = at^2$ and $Y = 2at$

$\boxed{Q7}$

$$\lim_{t\to 1}\begin{bmatrix}\frac{t^3-t}{t-1}\\[2mm] \frac{\sin(t-1)}{(t-1)}\end{bmatrix} = \begin{bmatrix}\lim_{t\to 1}\frac{t^3-t}{t-1}\\[2mm] \lim_{t\to 1}\frac{\sin(t-1)}{(t-1)}\end{bmatrix}$$

L'Hôpital

$$= \begin{bmatrix}\lim_{t\to 1}\frac{3t^2-1}{1}\\[2mm] \lim_{t\to 1}\frac{\cos(t-1)}{1}\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

$\boxed{Q8}$

$$\gamma(t) = \begin{bmatrix}t^3-t\\ (3t+5)^2\\ t^2+1\end{bmatrix} \quad \forall\, t\in\mathbb{R}$$

(a) $$\gamma(-1) = \begin{bmatrix}(-1)^3-(-1)\\ (-3+5)^2\\ 1+1\end{bmatrix} = \begin{bmatrix}0\\ 4\\ 2\end{bmatrix}$$

(b) $$\frac{d\gamma}{dt}(t) = \begin{bmatrix}3t^2-1\\ 2(3t+5)(3)\\ 2t\end{bmatrix} \Rightarrow \frac{d\gamma}{dt}(-1) = \begin{bmatrix}2\\ 12\\ -4\end{bmatrix}$$

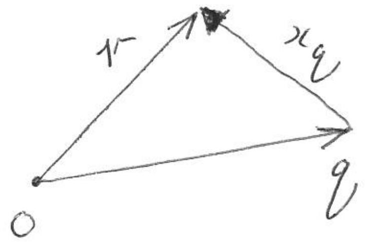
(c) Tangent line

$$\ell:\mathbb{R}\to\mathbb{R}^3 : s\mapsto \ell(s) = \gamma(-1)+s\frac{d\gamma}{dt}(-1)$$
$$= \begin{bmatrix}0\\ 4\\ 2\end{bmatrix} + s\begin{bmatrix}2\\ 12\\ -4\end{bmatrix}$$

Q9 Position $\gamma(t) = \begin{bmatrix} t^3 - t \\ (3t+5)^2 \\ t^2+1 \end{bmatrix}$

Velocity $\dfrac{d\gamma}{dt}(t) = \begin{bmatrix} 3t^2 - 1 \\ 2(3t+5)3 \\ 2t \end{bmatrix}$, Acceleration $\dfrac{d^2\gamma}{dt^2} = \begin{bmatrix} 6t \\ 18 \\ 2 \end{bmatrix}$

Q10 $x_q = r - q$



The principle of angular momentum

$$\frac{dA_q}{dt} = x_q \times F$$

$$\iff \frac{d}{dt}(x_q \times M) = x_q \times F$$ since $A_q = x_q \times M$

$$\iff \underbrace{\dot{x}_q \times m\dot{x}_q}_{= 0} + x_q \times \frac{dM}{dt} = x_q \times F$$ since $\dot{x}_q = \dot{r} = v$ and $M = mv$

$$\iff x_q \times \left(\frac{dM}{dt} - F\right) = 0$$

$$\iff \left(\frac{dM}{dt} - F\right) = 0$$ since q and, therefore, x_q is arbitrary

Newtons Second Law