MS 221 — Homework Set (4)

(Partial Derivatives and The Chain Rule)

QUESTION 1

In the case of the function $f(x,\, y,\, z) = x^2y - xy^2z + z^3$, and the point $p = (x,\, y,\, z)$, calculate the following:

$$\frac{\partial f}{\partial x}(p),\quad \frac{\partial f}{\partial y}(p),\quad \frac{\partial f}{\partial z}(p),\quad \frac{\partial^2 f}{\partial x^2}(p),\quad \frac{\partial^2 f}{\partial x \partial y}(p) \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}(p)$$

QUESTION 2

If f is again the function given in Question 1 calculate

$$\frac{\partial f}{\partial y}(p) \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}(p) \quad \text{where the point} \quad p = (-1,\, 0,\, 3)$$

QUESTION 3

Given that $\dfrac{d}{du}\tan^{-1} u = \dfrac{1}{1+u^2}$ calculate

$$\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)$$

QUESTION 4

Consider the function

$$f(x,\, y) = \frac{xy}{x^2+y^2} \quad \text{defined for all} \quad (x,\, y) \neq (0,\, 0).$$

In each of the following, investigate the behaviour of $f(p)$ as p approaches the origin:

(a) along the line $y = 2x$

(b) along the line $y = 3x$

(c) along any line $y = mx$.

What can be said about the existence or otherwise of $\lim_{p \to 0} f(p)$?

QUESTION 5

Consider the function

$$f(x,\,y)=\frac{x^2y}{x^4+y^2}\quad\text{defined for all}\quad (x,\,y)\neq(0,\,0).$$

In each of the following, investigate the behaviour of $f(p)$ as p approaches the origin:

(a) along any line $y=mx$

(b) along any parabola $y=mx^2$

What can be said about the existence or otherwise of $\lim_{p\to 0} f(p)$?

QUESTION 6

In the case of differentiable maps

$$\gamma:\boldsymbol{R}\to\boldsymbol{R}^3:t\mapsto\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}\quad\text{and}\quad f:\boldsymbol{R}^3\to\boldsymbol{R}:\begin{bmatrix}x\\y\\z\end{bmatrix}\mapsto f(x,\,y,\,z)$$

express the derivative $\dfrac{d}{dt}f(\,x(t),\,y(t),\,z(t)\,)$ in terms of the **Chain Rule.**

QUESTION 7

Let the point $\boldsymbol{p}$ and the curve γ be given by

$$\boldsymbol{p}=\begin{bmatrix}-3\\1\\2\end{bmatrix}\quad\text{and}\quad\gamma(t)=\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}=\begin{bmatrix}t^2-4\\t\\t^3+1\end{bmatrix}\quad\forall\ t\in\boldsymbol{R}.$$

If the map $f:\boldsymbol{R}^3\to\boldsymbol{R}:(x,\,y,\,z)\mapsto f(x,\,y,\,z)$ satisfies

$$\frac{\partial f}{\partial x}(\boldsymbol{p})=-1,\quad\frac{\partial f}{\partial y}(\boldsymbol{p})=2,\quad\frac{\partial f}{\partial z}(\boldsymbol{p})=5.$$

calculate $\dfrac{d}{dt}f(\gamma(t))$ at $t=1$.

MS 221 Homework Set (4)

Q1 $f(x,y,z) = x^2 y - x y^2 z + z^3$

$\frac{\partial f}{\partial x}(p) = 2xy - y^2 z.$ $\frac{\partial f}{\partial y}(p) = x^2 - 2xyz.$

$\frac{\partial f}{\partial z}(p) = xy^2 + 3z^2.$ $\frac{\partial^2 f}{\partial x^2}(p) = 2y$

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial}{\partial x}\left(x^2 - 2xyz\right) = 2x - 2yz.$$

$$\frac{\partial^2 f}{\partial y \partial x}(p) = \frac{\partial}{\partial y}\left(2xy - y^2 z\right) = 2x - 2yz.$$

Q2 When $p = (-1, 0, 3)$ in Q1 we get

$$\frac{\partial f}{\partial y}(p) = \left(x^2 - 2xyz\right)_{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}} = (-1)^2 - 0 = 1.$$

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \left(2x - 2yz\right)_{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}} = \left(-2 - 0\right) = -2.$$

$\boxed{Q3}$

$$\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{d}{du}\tan^{-1}u\right)\frac{\partial u}{\partial x}$$

By The Chain Rule with $u(x,y) = \frac{y}{x}$

$$= \left(\frac{1}{1+u^2}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$= \left(\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}.$$

$\boxed{Q4}$

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \forall\ (x,y) \neq (0,0)$$

(a) limit of $f(x,y)$ as $(x,y) \to 0$ along $y = 2x$

$$= \lim_{x\to 0} f(x, 2x) = \lim_{x\to 0}\frac{x.(2x)}{x^2+(2x)^2}$$

$$= \lim_{x\to 0}\frac{2x^2}{5x^2} = \frac{2}{5}.$$

Q4

(b) Here we seek $\lim_{x \to 0} f(x, 3x)$

$$= \lim_{x \to 0} \frac{x.(3x)}{x^2 + (3x)^2} = \lim_{x \to 0} \frac{3x^2}{10x^2}$$

$$= \frac{3}{10}$$

(c) $$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = mx}} \frac{xy}{x^2+y^2} = \lim_{x \to 0} \frac{x(mx)}{x^2 + (mx)^2}$$

$$= \lim_{x \to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}.$$

Since this quantity varies with the (slope of the) line (m) it follows that

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$$ does NOT exist!

Q5 $f(x,y) = \dfrac{x^2 y}{x^4 + y^2} \quad \forall\ (x,y) \neq (0,0)$

(a) Along the line $y = mx$ we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \left.\frac{x^2 y}{x^4+y^2}\right|_{y=mx}$$

$$= \lim_{x\to 0} \frac{m x^3}{x^4 + m^2 x^2}$$

$$= \lim_{x\to 0} \frac{mx}{x^2+m^2} = \frac{0}{0+m^2} = 0$$

$\forall\, m \neq 0$

along $y = 0$, $f(x,y) = 0$.

(b) along the parabola $y = mx^2$ we have

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx^2}} f(x,y) = \lim_{x\to 0} f(x, mx^2)$$

$$= \lim_{x\to 0} \left.\frac{x^2 y}{x^4+y^2}\right|_{y=mx^2}$$

$$= \lim_{x\to 0} \frac{m x^4}{x^4 + m^2 x^4}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}.$$

Thus $\lim_{p\to 0} f(p)$ does <u>NOT</u> exist even though $f(p) \to 0$ as $p \to 0$ along every line.

Q6

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}(p)\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(p)\frac{dy}{dt}(t) + \frac{\partial f}{\partial z}(p)\frac{dz}{dt}(t)$$

where $p = \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

We often abbreviate this to

$$\frac{d}{dt} f(x, y, z) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Q7

Note that $p = \gamma(1)$

$$\Rightarrow \left.\frac{d}{dt} f(\gamma(t))\right|_{t=1} = \frac{\partial f}{\partial x}(p)\frac{dx}{dt}(1) + \frac{\partial f}{\partial y}(p)\frac{dy}{dt}(1) + \frac{\partial f}{\partial z}(p)\frac{dz}{dt}(1)$$

$$= \Big[(-1)\,2t + (2)\,1 + (5)\,3t^2\Big]_{t=1}$$

$$= -2 + 2 + 15$$

$$= 15$$