MS 221 — Homework Set (5)

(Applications of The Chain Rule)

QUESTION 1

A particle moving on a plane has Cartesian and polar coordinates at time t given by $(x(t),\, y(t))$ and $(r(t),\, \theta(t))$, respectively. Thus,

$$x(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad y(t) = r(t)\sin\theta(t) \quad \forall\ t \in \boldsymbol{R}.$$

If the speed in Cartesian coordinates is given by $\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$ find the corresponding formula for the speed in terms of polar coordinates, that is, in terms of r, θ, $\dot{r}$ and $\dot{\theta}$.

QUESTION 2

A disc with **centre at the origin** rotates anti-clockwise with **constant angular speed** ω revolutions/sec about the origin. An insect on this disc is crawling in a straight line (relative to the disc) towards the centre at a constant speed (relative to the disc) of α cm/sec. If the polar coordinates of the insect at time $t = 0$ are $r(0) = 100$ cm and $\theta(0) = 0$ radians do the following:

(a) Find the polar coordinates $(r(t),\, \theta(t))$ of the insect at any subsequent time t.

(b) Use part (a) to determine the Cartesian coordinates $(x(t),\, y(t))$ of the insect at any subsequent time t.

(c) Find the velocity and acceleration (vectors) in Cartesian coordinates of the insect at any subsequent time t.

QUESTION 3

A function $f : \boldsymbol{R}^2 \to \boldsymbol{R} : (x,\, y) \mapsto f(x,\, y)$ satisfies

$$\frac{\partial f}{\partial x}(0,\, 0) = 3 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,\, 0) = -5.$$

If in addition, $f(ta,\, tb) = tf(a,\, b)$ for every $t \in \boldsymbol{R}$ and for every $(a,\, b) \in \boldsymbol{R}^2$, find $f(a,\, b)\ \forall\, (a,\, b) \in \boldsymbol{R}^2$.

Hint: $tf(a,\, b) \equiv f(ta,\, tb) \implies f(a,\, b) \equiv \dfrac{d}{dt} f(ta,\, tb).$

QUESTION 4

Express the partial derivative $\dfrac{\partial}{\partial x} f(\, u(x,\, y),\, v(x,\, y),\, w(x,\, y)\,)$ in terms of the **Chain Rule**.

QUESTION 5

Throughout this question Ω will denote the set in the xy-plane given by:

$$\Omega = \{ (x, y) \in \boldsymbol{R}^2 \mid y > 0 \}.$$

If the functions $\xi : \Omega \to \boldsymbol{R}$ and $\eta : \Omega \to \boldsymbol{R}$ are specified by

$$\xi(x, y) = x \ln y \quad \text{and} \quad \eta(x, y) = x,$$

express the partial differential equation

$$x \frac{\partial u}{\partial x} - y \ln y \frac{\partial u}{\partial y} = u \qquad \text{on } \Omega$$

as a partial differential equation in the (ξ, η) - coordinates and, hence or otherwise, solve this partial differential equation subject to the condition that

$$u(x, e) \equiv xe^x \quad \text{for all } x \in \boldsymbol{R}$$

QUESTION 6

Notation: In the case where $\omega = f(u(x, t), v(x, t))$ we will write $\dfrac{\partial \omega}{\partial u} := \dfrac{\partial f}{\partial u}(u, v)$,

$$\frac{\partial \omega}{\partial v} := \frac{\partial f}{\partial v}(u, v), \qquad \frac{\partial \omega}{\partial x} := \frac{\partial f}{\partial x}(u(x, t), v(x, t)), \qquad \frac{\partial \omega}{\partial t} := \frac{\partial f}{\partial t}(u(x, t), v(x, t))$$

and similarly for higher order derivatives. Now, if

$$u(x, t) = x + ct \quad \text{and} \quad v(x, t) = x - ct,$$

where c is a non-zero constant, show that

$$\frac{\partial^2 \omega}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \omega}{\partial t^2} \equiv 4 \frac{\partial^2 \omega}{\partial u \partial v}$$

QUESTION 7

Find all solutions of the (partial differential) equation $\dfrac{\partial^2 \omega}{\partial u \partial v} \equiv 0.$

Hint: If a function $h(r, s)$ satisfies $\dfrac{\partial h}{\partial r} \equiv 0$, then h is constant in r. That is, h **is a function of** s **only**.

QUESTION 8

Use Questions 6 and 7 above to show that **every solution** ω of the 1-dimensional wave equation

$$\frac{\partial^2 \omega}{\partial x^2} \equiv \frac{1}{c^2} \frac{\partial^2 \omega}{\partial t^2}$$

is of the form $\omega = \varphi(x + ct) + \psi(x - ct)$ where φ and ψ are arbitrary smooth functions

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Q1

Write $\dot{x}(t)$ for $\frac{dx}{dt}(t)$ etc.

$$\boxed{\begin{aligned} x(t) &= r(t)\cos\theta(t) \\ y(t) &= r(t)\sin\theta(t) \end{aligned}} \Rightarrow \boxed{\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}}$$

$$\Rightarrow (\dot{x}^2 + \dot{y}^2) = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2$$

$$= \left(\dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta\right) + \left(\dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\cos^2\theta\right)$$

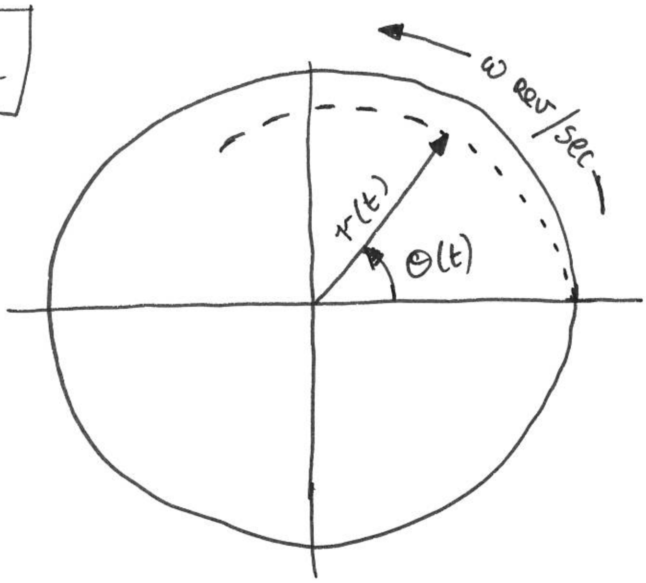
$$= \dot{r}^2(\cos^2\theta + \sin^2\theta) + 0 + r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

$$\Rightarrow \text{speed at time } t = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

Q2



(a) Position of insect at time t in polar coordinates is $(r(t), \theta(t))$

where

$$r(t) = 100 - \alpha t \text{ cm.}$$

$$\theta(t) = 2\pi\omega t \text{ (Radians)}$$

(b) From part (a)

$$x(t) = r(t)\cos\theta(t) = (100 - \alpha t)\cos 2\pi\omega t \text{ cm.}$$

$$y(t) = r(t)\sin\theta(t) = (100 - \alpha t)\sin 2\pi\omega t \text{ cm.}$$

(c)

$$\text{velocity} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\alpha\cos 2\pi\omega t - 2\pi\omega(100 - \alpha t)\sin 2\pi\omega t \\ -\alpha\sin 2\pi\omega t + 2\pi\omega(100 - \alpha t)\cos 2\pi\omega t \end{bmatrix}$$

$$\text{acceleration} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 2\alpha(2\pi\omega)\sin 2\pi\omega t - (2\pi\omega)^2(100 - \alpha t)\cos 2\pi\omega t \\ -2\alpha(2\pi\omega)\cos 2\pi\omega t - (2\pi\omega)^2(100 - \alpha t)\sin 2\pi\omega t \end{bmatrix}$$

Q3 $\quad t f(a,b) \equiv f(ta, tb)$

$$\Rightarrow f(a,b) \equiv \frac{d}{dt} f(ta, tb)$$

$$\equiv \frac{d}{dt} f(x(t), y(t)) \qquad \begin{cases} x(t) = ta \\ y(t) = tb \end{cases}$$

$$\equiv \frac{\partial f}{\partial x}(p) \frac{dx}{dt} + \frac{\partial f}{\partial y}(p) \frac{dy}{dt} \qquad \begin{cases} \text{where} \\ p = (x(t), y(t)) \\ \quad = (ta, tb) \end{cases}$$

$$\equiv \frac{\partial f}{\partial x}(p) \cdot a + \frac{\partial f}{\partial y}(p)\, b$$

In particular when $t = 0$, $p = (0,0)$

$$= a \frac{\partial f}{\partial x}(0,0) + b \frac{\partial f}{\partial y}(0,0)$$

$$= 3a - 5b$$

Q4 $\quad \frac{\partial}{\partial x} f(u(x,y), v(x,y), w(x,y))$

$$= \frac{\partial f}{\partial u}(p) \frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(p) \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial w}(p) \frac{\partial w}{\partial x}(x,y)$$

where $p = (u(x,y), v(x,y), w(x,y))$

$\boxed{Q5}$ $\boxed{\xi(x,y) = x \ln y}$ $\boxed{\eta(x,y) = x}$

$$x \frac{\partial u}{\partial x} - y \ln y \frac{\partial u}{\partial y} = u \quad \text{on } \Omega$$

$$\Rightarrow x\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right] - y \ln y\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right] = u$$

$$\Rightarrow x\left[\cancel{\frac{\partial u}{\partial \xi}\cdot(\ln y)} + \frac{\partial u}{\partial \eta}\cdot(1)\right] - \cancel{y \ln y\left[\frac{\partial u}{\partial \xi}\cdot\left(\frac{x}{y}\right)\right.} + \left.\frac{\partial u}{\partial \eta}\cdot(0)\right] = u$$

$$\Rightarrow x \frac{\partial u}{\partial \eta} = u \quad \Rightarrow \boxed{\eta \frac{\partial u}{\partial \eta} = u}$$

$$\Rightarrow \int \frac{1}{u}\frac{\partial u}{\partial \eta}\, d\eta = \int \frac{1}{\eta}\, d\eta$$

$$\Rightarrow \int \frac{1}{u}\, du = \int \frac{1}{\eta}\, d\eta \quad \text{(integration by substitution)}$$

$$\Rightarrow \ln u = \ln \eta + \ln \varphi(\xi) = \ln \eta \varphi(\xi)$$

$$\Rightarrow u(\xi, \eta) = \eta \varphi(\xi)$$

$$\Rightarrow u(x,y) := u(\xi(x,y), \eta(x,y)) = x \cdot \varphi(x \ln y)$$

$$x e^x \equiv u(x,e) = x \varphi(x \cdot 1) \Rightarrow \boxed{\varphi(x) = e^x}$$

$$\Rightarrow u(x,y) = x e^{x \ln y} = x e^{\ln y^x} = x y^x$$

$\boxed{Q6}$ $\boxed{u(x,t) = x + ct}$ $\boxed{v(x,t) = x - ct}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t}$$

$$\Rightarrow \quad w_t = w_u \cdot (c) + w_v \cdot (-c)$$

write $w_t = \frac{\partial w}{\partial t}$, $w_u = \frac{\partial w}{\partial u}$ etc

$$\Rightarrow \quad w_t = c\left[w_u - w_v\right]$$

$$\Rightarrow \quad w_{tt} = c\left[\frac{\partial w_u}{\partial t} - \frac{\partial w_v}{\partial t}\right]$$

$$= c\left[\frac{\partial w_u}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w_u}{\partial v}\frac{\partial v}{\partial t} - \left(\frac{\partial w_v}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w_v}{\partial v}\frac{\partial v}{\partial t}\right)\right]$$

$$= c\left[w_{uu}\cdot(c) + w_{uv}\cdot(-c) - \left(w_{vu}\cdot(c) + w_{vv}\cdot(-c)\right)\right]$$

$$= c^2\left[w_{uv} - 2w_{uv} + w_{vv}\right]$$

$$\Rightarrow \quad \boxed{w_{tt} = c^2\left[w_{uv} - 2w_{uv} + w_{vv}\right.} \text{ ----- } (1)$$

Similarly

$$w_x = w_u + w_v$$

and

$$\boxed{w_{xx} = w_{uu} + 2w_{uv} + w_{vv}} \text{ --- } (2)$$

$$\Rightarrow \quad w_{xx} - \frac{1}{c^2}w_{tt} = 4\,w_{uv} \text{ ----- } (2) - \frac{1}{c^2}(1)$$

Q7

$$\frac{\partial^2 w}{\partial u \partial v} = 0$$

i.e. $$\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = 0$$

$$\Rightarrow \quad \frac{\partial w}{\partial v} = f_1(v)$$

$$\Rightarrow \quad w = \int f_1(v)\,dv + f_2(u)$$

$$\Rightarrow \quad w = \phi(v) + \phi(u)$$

arbitrary smooth functions

Q8

$u = x + ct$
$v = x - ct$ in Q5

and

$$4\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0$$

wave equ^n

$$\Rightarrow \quad 4\frac{\partial^2 w}{\partial u \partial v} = 0$$

$$\Rightarrow \quad w = \phi(u) + \psi(v)$$ by Q6

$$\Rightarrow \quad w = \phi(x+ct) + \psi(x-ct)$$