# MS 221 — Homework Set (6)

## (The Chain Rule and The Gradient)

### QUESTION 1

Let $\boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and let the function $f : \boldsymbol{R}^3 \to \boldsymbol{R} : \begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto f(u, v, w)$ satisfy

$$\frac{\partial f}{\partial u}(\boldsymbol{p}) = -3, \quad \frac{\partial f}{\partial v}(\boldsymbol{p}) = 2, \quad \frac{\partial f}{\partial w}(\boldsymbol{p}) = 5.$$

If the functions $u, v, w : \boldsymbol{R}^2 \to \boldsymbol{R}$ are defined by

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= x + xy \\ w(x,y) &= 2 + xy - y^3 \end{aligned}$$

calculate

$$\frac{\partial}{\partial x} f(u(x,y), v(x,y), w(x,y)) \quad \text{at} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

### QUESTION 2

Find $\nabla f_{\boldsymbol{p}}$ where the function $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$f(x,\, y,\, z) = x^2 z + y \ln(z^2 + 1) \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

### QUESTION 3

Find all points $\boldsymbol{p} \in \boldsymbol{R}^3$ such that $\nabla \varphi_{\boldsymbol{p}} = \boldsymbol{0}$ where the function $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ is given by

$$\varphi(x,\, y,\, z) = x^2 y + y^2 z + z^2 x.$$

## QUESTION 4

Let $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ be as in Question 3 and let $M$ be the **level set** determined by

$$\varphi(x,\, y,\, z) = 1.$$

Now, show that the point $\boldsymbol{p} = (1,\, -1,\, 1)$ is on the level set $M$ and find the equation of the **tangent plane** to $M$ at $\boldsymbol{p}$.

## QUESTION 5

The equation $z = f(x,\, y)$ defines a surface $M$ in $\boldsymbol{R}^3$. If we put $z_0 = f(x_0,\, y_0)$, then the point $\boldsymbol{p}_0 = (x_0,\, y_0,\, z_0)$ is on the surface $M$. Now, find the equation of the **tangent plane** to $M$ at $\boldsymbol{p}_0$.

**Hint:** Define the function $\varphi(x,\, y,\, z) \equiv z - f(x,\, y)$, then

$$\boxed{z = f(x,\, y)} \quad \Longleftrightarrow \quad \boxed{\varphi(x,\, y,\, z) = 0}$$

## QUESTION 6

Let the functions $\varphi,\, \psi : \boldsymbol{R}^2 \to \boldsymbol{R}$ be given by:

$$\varphi(x,\, y) = x^2 - y^2 + x \qquad \text{and} \qquad \psi(x,\, y) = 2xy + y$$

If $\mathcal{C}_1,\, \mathcal{C}_2 \subset \boldsymbol{R}^2$ denote the **level sets** (i.e. curves) defined by

$$\varphi(x,\, y) \equiv 0 \qquad \text{and} \qquad \psi(x,\, y) \equiv 0$$

respectively, show that $(0,\, 0) \in \mathcal{C}_1 \cap \mathcal{C}_2$ and find the **angle of intersection** of $\mathcal{C}_1$ and $\mathcal{C}_2$ at this point.

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**Hint:** Define the function $\varphi(x, y, z) \equiv z - f(x, y)$, then

$$\boxed{z = f(x, y)} \iff \boxed{\varphi(x, y, z) = 0}$$

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If $\mathcal{C}_1, \mathcal{C}_2 \subset \boldsymbol{R}^2$ denote the **level sets** (i.e. curves) defined by

$$\varphi(x, y) \equiv 0 \quad \text{and} \quad \psi(x, y) \equiv 0$$

respectively, show that $(0, 0) \in \mathcal{C}_1 \cap \mathcal{C}_2$ and find the **angle of intersection** of $\mathcal{C}_1$ and $\mathcal{C}_2$ at this point.

## QUESTION 7

Determine the **shortest distance** from the point $(0, b)$ on the $y$-axis to the parabola $x^2 - 4y = 0$ in each of the following ways:

(i) Use the method of **Lagrange multipliers**.

(ii) Use the constraint $x^2 - 4y = 0$ to **eliminate one of the variables**, thus reducing the problem to the calculus of one variable.

**Hint:** $\boxed{\text{Distance is minimized}} \iff \boxed{(\text{Distance})^2 \text{ is minimized}}$

## QUESTION 8

Let $\wp$ be the plane in $\boldsymbol{R}^3$ which passes through the point $\boldsymbol{p}$ and is normal to the vector $\boldsymbol{n}$. If $\boldsymbol{q}$ is any point in $\boldsymbol{R}^3$, use the method of **Lagrange multipliers** to find the shortest distance from the point $\boldsymbol{q}$ to the plane $\wp$.

## QUESTION 9

The cone $z^2 = x^2 + y^2$ is cut by the plane $2x + 2y + 2z = 4$ in a curve $\mathcal{C}$. Find the points on $\mathcal{C}$ which are nearest and furthest away from the $xy$-plane.

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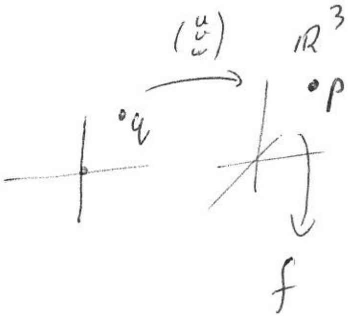
**Q1** With $q = (-1, 0)$ we have $p = \begin{bmatrix} u(q) \\ v(q) \\ w(q) \end{bmatrix}$

$$\frac{\partial}{\partial x} f(u,v,w) = \frac{\partial f}{\partial u}(p)\,\frac{\partial u}{\partial x}(q) + \frac{\partial f}{\partial v}(p)\,\frac{\partial v}{\partial x}(q) + \frac{\partial f}{\partial w}(p)\,\frac{\partial w}{\partial x}(q)$$




$$= \Big[(-3)\,2x + (2)(1+y) + (5)\,y\Big]_{q = \binom{x}{y} = \binom{-1}{0}}$$

$$= \big[(-3)(-2) + (2).1 + (5).0\big]$$

$$= 8$$

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**Q2** $f(x,y,z) = x^2 z + y \ln(z^2+1)$

$$\nabla f_p = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}_p = \begin{bmatrix} 2xz \\ \ln(z^2+1) \\ x^2 + \frac{2yz}{z^2+1} \end{bmatrix}_{p = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

**Q3**

$$\phi(x,y,z) = x^2 y + y^2 z + z^2 x$$

$$\nabla\phi = \begin{bmatrix}\phi_x\\ \phi_y\\ \phi_z\end{bmatrix} = \begin{bmatrix}2xy + z^2\\ 2yz + x^2\\ 2zx + y^2\end{bmatrix} \quad \text{now } \nabla\phi = 0$$

$$\Longleftrightarrow \boxed{\begin{aligned}2xy + z^2 &= 0\\ 2yz + x^2 &= 0\\ 2zx + y^2 &= 0\end{aligned}} \quad \begin{matrix}\text{or } x=0\\ \text{or } y=0\\ \text{or } z=0\\ \Longleftrightarrow\end{matrix} \quad \boxed{\begin{aligned}2xyz + z^3 &= 0\\ 2xyz + x^3 &= 0\\ 2xyz + y^3 &= 0\end{aligned}}$$

$\Longleftrightarrow$ either $x = y = z = 0$

OR $x^3 = y^3 = z^2$

and

$2x^2 + x^2 = 0$

$\Rightarrow x = 0$ so again $x = y = z = 0$

So $\nabla\phi_p = 0 \Longleftrightarrow p = \begin{bmatrix}0\\0\\0\end{bmatrix}$

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**Q4**

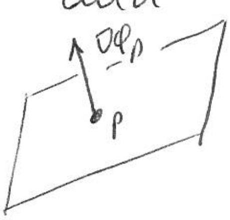
$p = (1,-1,1)$

$\phi(p) = -1 + 1 + 1 = 1 \Rightarrow \boxed{p \in M}$

a normal to $M$ at $p$ is $\nabla\phi_p = \begin{bmatrix}2xy + z^2\\ 2yz + x^2\\ 2zx + y^2\end{bmatrix}_p$

i.e. $\nabla\phi_p = \begin{bmatrix}-2+1\\ -2+1\\ 2+1\end{bmatrix} = \begin{bmatrix}-1\\-1\\3\end{bmatrix}$

and the tangent plane to $M$ at $p$ is




$$\left\langle \begin{pmatrix}x\\y\\z\end{pmatrix} - p \,,\, \nabla\phi_p \right\rangle = 0 \Longleftrightarrow \left\langle \begin{pmatrix}x-1\\y+1\\z-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\3\end{pmatrix}\right\rangle = 0$$

$$\Longleftrightarrow -(x-1) - (y+1) + 3(z-1) = 0$$

**Q5** $z = f(x,y) \iff \underbrace{z - f(x,y)}_{\phi(x,y,z)} \equiv 0$

$$\text{Normal} = \nabla\phi = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \quad \text{note } f_x = \frac{\partial f}{\partial x} \text{ etc.}$$

Equation of tangent plane is $\langle X - p_0, \nabla\phi_{p_0} \rangle = 0$

$$\Rightarrow \left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \right\rangle = 0$$

$$\Rightarrow f_x(p_0)(x - x_0) + f_y(p_0)(y - y_0) - (z - z_0) = 0$$

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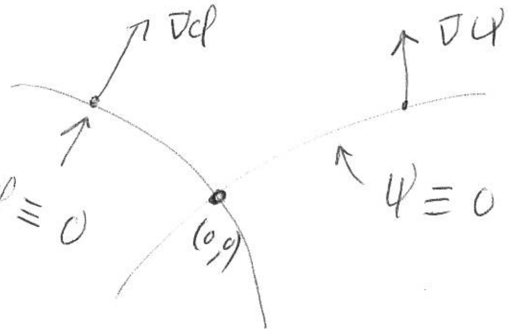
**Q6**




curves $\mathcal{C}_1$ & $\mathcal{C}_2$ intersect orthogonally at $(0,0)$

$$\iff \langle \nabla\phi, \nabla\psi \rangle_{(0,0)} = 0$$

Here $\nabla\phi = \begin{bmatrix} +2x+1 \\ -2y \end{bmatrix}$ $\quad \nabla\psi = \begin{bmatrix} 2y \\ 2x+1 \end{bmatrix}$

$$\langle \nabla\phi, \nabla\psi \rangle = 4xy + 2y - 4xy - 2y \equiv 0.$$