MS 221 — Homework Set (7)

(Lagrange Multipliers / Grad, Div and Curl)

QUESTION 1

Determine the **shortest distance** from the point $(0, b)$ on the y-axis to the parabola $x^2 - 4y = 0$ in each of the following ways:

(i) Use the method of **Lagrange multipliers**.

(ii) Use the constraint $x^2 - 4y = 0$ to **eliminate one of the variables**, thus reducing the problem to the calculus of one variable.

Hint: $\boxed{\text{Distance is minimized}} \iff \boxed{(\text{Distance})^2 \text{ is minimized}}$

QUESTION 2

Let $\wp$ be the plane in $\boldsymbol{R}^3$ which passes through the point $\boldsymbol{p}$ and is normal to the vector $\boldsymbol{n}$. If $\boldsymbol{q}$ is any point in $\boldsymbol{R}^3$, use the method of **Lagrange multipliers** to find the shortest distance from the point $\boldsymbol{q}$ to the plane $\wp$.

QUESTION 3

The cone $z^2 = x^2 + y^2$ is cut by the plane $2x + 2y + 2z = 4$ in a curve $\boldsymbol{C}$. Find the points on $\boldsymbol{C}$ which are nearest and furthest away from the xy-plane.

QUESTION 4

Use the method of **Lagrange multipliers** to find the points on the curve

$$3x^2 - 8xy - 3y^2 = 5$$

which are nearest and furthest away from the origin.

QUESTION 5

Calculate $\nabla\varphi_{\boldsymbol{p}}$ (that is, the **gradient** of φ at $\boldsymbol{p}$) where the function $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$\varphi(x,\, y,\, z) = x^2 z + e^{yz} \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{respectively.}$$

QUESTION 6

Calculate $\nabla .\, \boldsymbol{F}_{\boldsymbol{p}}$, (that is, the **divergence** of $\boldsymbol{F}$ at $\boldsymbol{p}$) where the vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$\boldsymbol{F}(x,\, y,\, z) = \begin{bmatrix} x^2 y \\ x - yz \\ \sin(yz) \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{respectively.}$$

QUESTION 7

Calculate $\nabla \times \boldsymbol{F}_{\boldsymbol{p}}$, (that is, the **curl** of $\boldsymbol{F}$ at $\boldsymbol{p}$) where the vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are as given in Question ~~4~~ 6

QUESTION 8

In the case of any (smooth) scalar field $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ and vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ establish the following

(i) $\nabla .\, (\varphi\, \boldsymbol{F}) = (\nabla\varphi) .\, \boldsymbol{F} + \varphi\, (\nabla .\, \boldsymbol{F})$.

(ii) $\nabla \times (\varphi\, \boldsymbol{F}) = (\nabla\varphi) \times \boldsymbol{F} + \varphi\, (\nabla \times \boldsymbol{F})$.

(iii) $\nabla \times (\nabla\varphi) \equiv \boldsymbol{0}$.

(iv) $\nabla .\, (\nabla \times \boldsymbol{F}) \equiv 0$.

(v) $\nabla .\, (\nabla\varphi) = \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial z^2}$.

QUESTION 9

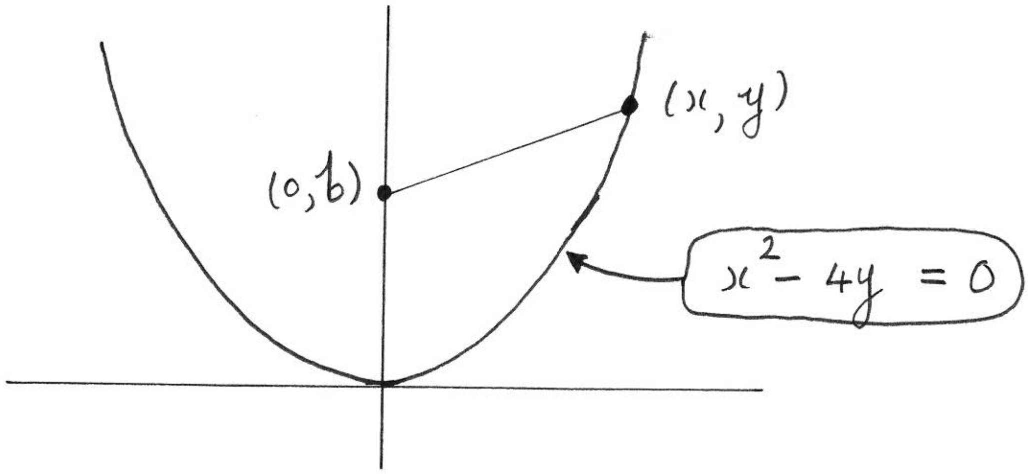
Use the **Chain Rule** to express the two dimensional **Laplacian**

$$\nabla .\, (\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

in terms of **polar coordinates**.

MS 221 HOMEWORK SET 7

Q1



Minimize $f(x,y) = (\text{distance})^2 = x^2 + (y-b)^2$

subject to the constraint

$$g(x,y) = x^2 - 4y = 0$$

(i) Lagrange Multipliers:

Solve $$\nabla f - \lambda \nabla g = 0$$

$$\iff \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} - \lambda \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 2x - \lambda 2x \\ 2(y-b) - \lambda(-4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow \boxed{\begin{array}{l} x(1-\lambda) = 0 \quad \ldots\ldots (1) \\ (y-b) + 2\lambda = 0 \quad \ldots\ldots (2) \end{array}}$$

Equation (1) results in the following two cases:

Case 1: $\boxed{x = 0} \xRightarrow{\text{constraint}} y = 0$

and $(\text{dist})^2 = f(0,0) = b^2$

Case 2: $\boxed{\lambda = 1} \xRightarrow{\text{Eq}^n (2)} y = (b-2)$

$\xRightarrow{\text{constraint}} x = \pm 2\sqrt{b-2}$

Note: For this case to arise we require $b \geqslant 2$ for $x \in \mathbb{R}$.

Here $(\text{dist})^2 = x^2 + (y-b)^2$

$= 4(b-2) + 4$

$= 4b - 4$

The question now is:

Is distance in case 1 $\leq$ distance in Case 2

This is true

$$\iff \quad b^2 \leq 4b-4$$

$$\iff \quad b^2-4b+4 \leq 0$$

$$\iff \quad (b-2)^2 \leq 0$$

and this is never true unless $b=2$, in which case we get equality.

Thus, shortest distance is always given by the formula in case 2. That is,

$$\text{distance} = 2\sqrt{b-1}$$

(ii) <u>Eliminate one of the variables using constraint:</u>

constraint $\quad x^2-4y=0$

$$\implies \quad x^2 = 4y$$

So $(\text{dist})^2 = x^2 + (y-b)^2$

$$= 4y + (y-b)^2 \quad \forall\ y \geq 0.$$

So we must minimize

$$\varphi : [0,\infty) \longrightarrow \mathbb{R} : y \longmapsto \underbrace{4y + (y-b)^2}_{\varphi(y)}$$

$$\varphi'(y) = 4 + 2(y-b) = 2y - 2(b-2)$$

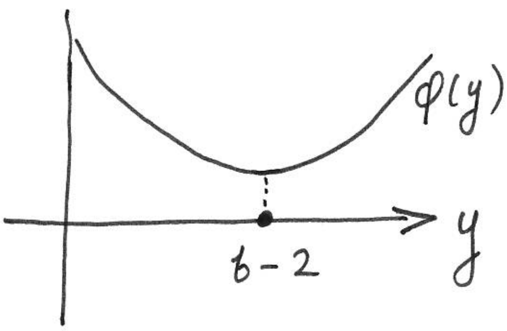
$$\Rightarrow \varphi'(y) \text{ is } \begin{cases} > 0 & \text{if } y > (b-2) \\ < 0 & \text{if } y < (b-2) \end{cases}$$

CASE (A): $\boxed{b \leq 2}$ $\Rightarrow \varphi'(y) \geq 0 \;\; \forall\, y \in [0,\infty)$

$\Rightarrow \varphi$ is increasing on $[0,\infty)$

$\Rightarrow$ minimum value of $\varphi = \varphi(0)$
$= b^2$

CASE (B): $b > 2$ $\Rightarrow \varphi'(y)$ is $\begin{cases} < 0 & \forall\, y \in [0, b-2) \\ > 0 & \forall\, y \in (b-2, \infty) \end{cases}$



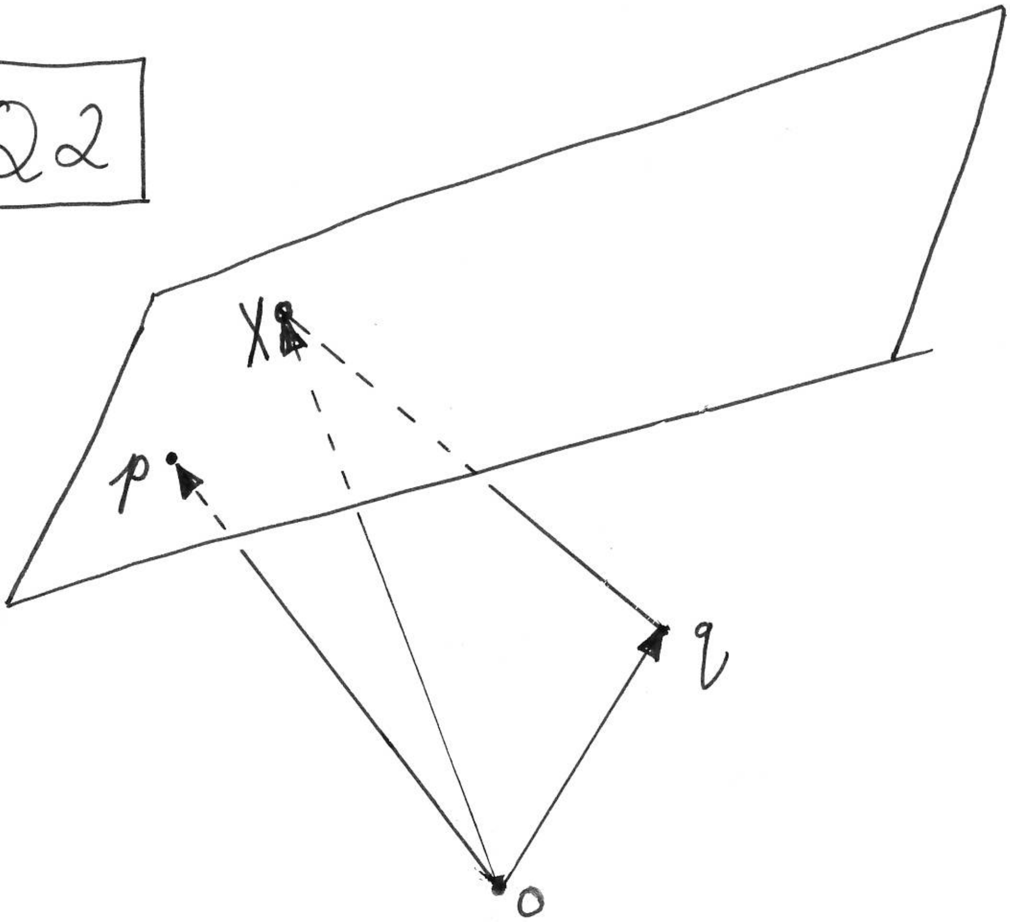
So minimum value of φ is

$= \varphi(b-2)$

$= 4(b-2) + (b-2-b)^2$

$= 4b - 4$

Q2



We must minimize $\| X - q \|^2$

subject to the constraint that $X \in$ plane

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad q = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Thus, the equation of the plane is

$$\langle X - p , n \rangle = 0$$

$$\Rightarrow \quad \langle X , n \rangle = \langle p , n \rangle$$

$$\Rightarrow \quad ax + by + cz = \underbrace{\quad}_{} d$$

Therefore, we must minimize

$$f(x,y,z) = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$$

subject to the constraint

$$g(x,y,z) = ax + by + cz = d.$$

Solve

$$\nabla f - \lambda \nabla g = 0$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} &: \quad 2(x-x_1) - \lambda a = 0 \ \ldots\ldots (1) \\ \frac{\partial}{\partial y} &: \quad 2(y-y_1) - \lambda b = 0 \ \ldots\ldots (2) \\ \frac{\partial}{\partial z} &: \quad 2(z-z_1) - \lambda c = 0 \ \ldots\ldots (3) \end{aligned}}$$

$$\Rightarrow (X - q) = \frac{\lambda}{2} n \ \ldots\ldots (4)$$

That is, the line joining q to X meets the plane orthogonally (i.e. is parallel to n). In any case, the shortest distance from q to the plane is

$$\| X - q \| = \left| \frac{\lambda}{2} \right| \| n \|.$$

The constraint

$$\langle X - p , n \rangle = 0$$

$$\Rightarrow \langle (X-q) + (q-p) , n \rangle = 0$$

$$\Rightarrow \langle X - q , n \rangle + \langle q - p , n \rangle = 0$$

$$\Rightarrow \langle X - q , n \rangle = \langle p - q , n \rangle .$$

Now use equation (4) to obtain

$$\left\langle \frac{\lambda}{2} n , n \right\rangle = \langle p - q , n \rangle$$

$$\| \qquad$$

$$\frac{\lambda}{2} \| n \|^2$$

$$\Rightarrow \frac{\lambda}{2} \| n \| = \frac{\langle p - q , n \rangle}{\| n \|}$$

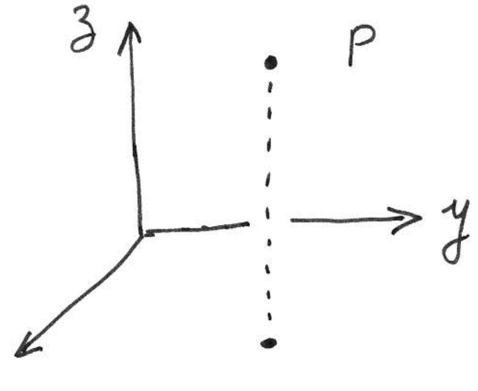
Thus the shortest distance

$$\| X - q \| = \left| \frac{\lambda}{2} \right| \| n \| = \frac{| \langle p - q , n \rangle |}{\| n \|}$$

Q3 again minimize $(\text{distance})^2$.

<u>Note</u>, the distance from a point

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ to the



xy-plane is $|z|$

$$\Rightarrow (\text{distance})^2 = z^2$$

Thus, we must minimize

$$f(x, y, z) = z^2$$

subject to the constraints

$$g_1(x, y, z) = z^2 - x^2 - y^2 = 0$$

$$g_2(x, y, z) = 2x + 2y + 2z = 4.$$

Solve $\nabla f - \lambda_1 \nabla g_1 - \lambda_2 \nabla g_2 = 0$

$$\frac{\partial}{\partial x}: \quad 0 - \lambda_1(-2x) - \lambda_2 2 = 0 \quad \ldots\ldots (1)$$

$$\frac{\partial}{\partial y}: \quad 0 - \lambda_1(-2y) - \lambda_2 2 = 0 \quad \ldots\ldots (2)$$

$$\frac{\partial}{\partial z}: \quad 2z - \lambda_1(2z) - \lambda_2 2 = 0 \quad \ldots\ldots (3)$$

$(Eq^n)_1 - (Eq^n)_2$

$$\Longrightarrow \quad 2x\lambda_1 - 2y\lambda_1 = 0$$

$$\Longrightarrow \quad (x-y)\lambda_1 = 0 \;\ldots\ldots (4)$$

So that two cases arise:

CASE 1 $\boxed{\lambda_1 = 0}$ Then

$$\overset{(1)}{\Longrightarrow} \lambda_2 = 0 \quad \text{and} \quad \overset{(3)}{\Longrightarrow} z = \lambda_1 z - \lambda_2 = 0$$

That is, $\boxed{z = 0}$. But then the constraint

$$g_1(x,y,z) = 0 \Longrightarrow x^2 + y^2 = 0$$

$$\Longrightarrow x = y = 0$$

So $\boxed{\lambda_1 = 0} \Longrightarrow \boxed{x = y = z = 0}$

But this is impossible because from the constraint $g_2(x,y,z) = 4$ we have

$$x + y + z = 2$$

Therefore, CASE 1 is NOT valid.

CASE 2: $\boxed{x = y}$

The constraints $\boxed{\begin{aligned} g_1(x, y, z) &= 0 \\ g_2(x, y, z) &= 4 \end{aligned}}$ now

imply $\boxed{\begin{aligned} z^2 - 2x^2 &= 0 \\ 2x + z &= 2 \end{aligned}}$

$\Rightarrow$ $\boxed{\begin{aligned} z^2 &= 2x^2 \\ z^2 &= \left[2(1-x)\right]^2 \end{aligned}}$

$\Rightarrow$ $4\left[x^2 - 2x + 1\right] = 2x^2$

$\Rightarrow$ $x^2 - 4x + 2 = 0$

$\Rightarrow$ $x = 2 \pm \sqrt{2}$

and $z^2 = x^2 + y^2 = 2x^2$

$= 2(2 \pm \sqrt{2})^2 = 2\left[4 \pm 2\sqrt{2} + 2\right]$

$= 2\left[6 \pm 2\sqrt{2}\right]$

For "nearest" take "$-$" here and farthest take $+$

$\boxed{Q4}$ again minimize/maximize $(\text{distance})^2$

So minimize

$$f(x,y,z) = x^2 + y^2$$

subject to the constraint that

$$g(x,y,z) = 3x^2 - 8xy - 3y^2 = 5$$

Solve $\nabla f - \lambda \nabla g = 0$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x}: &\quad 2x - \lambda(6x - 8y) = 0 \;.....(1) \\ \frac{\partial}{\partial y}: &\quad 2y - \lambda(-8x - 6y) = 0 \;.....(2) \end{aligned}}$$

$$\xRightarrow{(1)\times y} \quad 2xy - \lambda\left(6xy - 8y^2\right) = 0 \;....(3)$$

$$\xRightarrow{(2)\times x} \quad 2xy + \lambda\left(8x^2 + 6xy\right) = 0 \;...(4)$$

$$\xRightarrow{(4)-(3)} \quad \lambda\left(8x^2 + 12xy - 8y^2\right) = 0 \;....(5)$$

From Equation (5) two cases arise:

Case 1: $\boxed{\lambda = 0}$ Here, it follows from Equations (1) & (2) that,

$$x = y = 0.$$

But now the constraint $g(x, y, z) = 5$ is NOT satisfied. Thus Case 1 does NOT hold.

Case 2: $8x^2 + 12xy - 8y^2 = 0$

$$\Rightarrow 2x^2 + 3xy - 2y^2 = 0 \quad \text{......} (6)$$

$$\boxed{\begin{aligned} &\Rightarrow 6x^2 + 9xy - 6y^2 = 0 \\ \text{constraint } &\Rightarrow 6x^2 - 16xy - 6y^2 = 10 \end{aligned}}$$

$$\Rightarrow 25xy = -10$$

$$\Rightarrow y = -\frac{2}{5x} \quad \text{......} (7)$$

$$(6) \,\&\, (7) \Longrightarrow 2x^2 + 3x\left(-\frac{2}{5x}\right) - 2\left(-\frac{2}{5x}\right)^2 = 0$$

$$\Longrightarrow 25x^4 - 15x^2 - 4 = 0$$

$$\Rightarrow (5x^2-4)\underbrace{(5x^2+1)}_{\vee\, 0} = 0$$

$$\Rightarrow \quad 5x^2 - 4 = 0$$

$$\Rightarrow \quad x = \pm \frac{2}{\sqrt{5}}$$

and $\quad y = -\frac{2}{5}\cdot\frac{1}{x} = -\frac{2}{5}\cdot\left(\pm\frac{\sqrt{5}}{2}\right) = \mp\frac{1}{\sqrt{5}}$

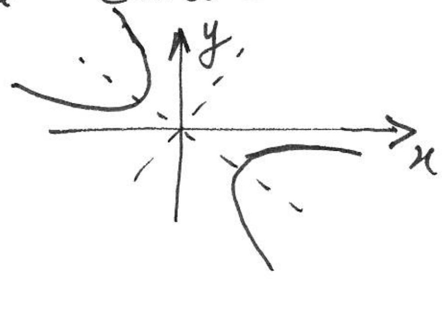
$$\Rightarrow (x,y) = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \text{ or } \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

In either case

$$f(x,y) = x^2 + y^2$$

$$= \frac{4}{5} + \frac{1}{5} = 1$$

<u>note</u>: Both minimize distance, there is no maximum distance since curve $3x^2 - 8xy - 3y^2 = 5$ looks like



$\boxed{Q5}$ $\phi(x, y, z) = x^2 z + e^{yz}$, $p = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

$$\nabla\phi_p = \begin{bmatrix} \frac{\partial\phi}{\partial x}(p) \\ \frac{\partial\phi}{\partial y}(p) \\ \frac{\partial\phi}{\partial z}(p) \end{bmatrix} = \begin{bmatrix} 2xz \\ z e^{yz} \\ x^2 + y e^{yz} \end{bmatrix}_p = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$\boxed{Q6}$

$$\nabla . F_p = \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right]_p$$

$$= \left[\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(x - yz) + \frac{\partial}{\partial z} \sin(yz) \right]_{p = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}$$

$$= \left[2xy - z + y\cos(yz) \right]_{p = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}$$

$$= \left[-6 - 0 + 3 \right]$$

$$= -3.$$

Q7

$$\nabla \times F_p = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}_p$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & (x - yz) & \sin(yz) \end{vmatrix}_{p = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} z\cos(yz) + y \\ 0 - 0 \\ 1 - x^2 \end{bmatrix}_{p = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Q8

(i) $$\nabla \cdot (\varphi F) = \frac{\partial}{\partial x}(\varphi F_1) + \frac{\partial}{\partial y}(\varphi F_2) + \frac{\partial}{\partial z}(\varphi F_3)$$

$$= \left[\left(\frac{\partial \varphi}{\partial x}\right) F_1 + \varphi \frac{\partial F_1}{\partial x} \right] + \left[\left(\frac{\partial \varphi}{\partial y}\right) F_2 + \varphi \frac{\partial F_2}{\partial y} \right] + \left[\cdots \right]$$

DOT - PRODUCT

$$= \left[\left(\frac{\partial \varphi}{\partial x}\right) F_1 + \left(\frac{\partial \varphi}{\partial y}\right) F_2 + \left(\frac{\partial \varphi}{\partial z}\right) F_3 \right] + \varphi \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right]$$

$$= (\nabla \varphi) \cdot F + \varphi (\nabla \cdot F)$$

(ii) $\nabla \times (\varphi F) = \begin{bmatrix} \frac{\partial}{\partial y}(\varphi F_3) - \frac{\partial}{\partial z}(\varphi F_2) \\ \vdots \end{bmatrix}$

2nd & 3rd components are similar

$$= \begin{bmatrix} \left(\frac{\partial \varphi}{\partial y}\right) F_3 + \varphi \frac{\partial F_3}{\partial y} - \left(\frac{\partial \varphi}{\partial z}\right) F_2 - \varphi \frac{\partial F_2}{\partial z} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial \varphi}{\partial y}\right) F_3 - \left(\frac{\partial \varphi}{\partial z}\right) F_2 \\ \vdots \end{bmatrix} + \varphi \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \vdots \end{bmatrix}$$

$$= (\nabla \varphi) \times F + \varphi \nabla \times F$$

(iii) $\nabla \times (\nabla \varphi) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \begin{bmatrix} \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \\ \vdots \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ Because of the equality of mixed partial derivatives.

(iv) $\nabla \cdot (\nabla \times F) = \frac{\partial}{\partial x}(\nabla \times F)_1 + \frac{\partial}{\partial y}(\nabla \times F)_2 + \frac{\partial}{\partial z}(\nabla \times F)_3$

$$= \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\cdots\right] + \frac{\partial}{\partial z}\left[\cdots\right]$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cdots$$

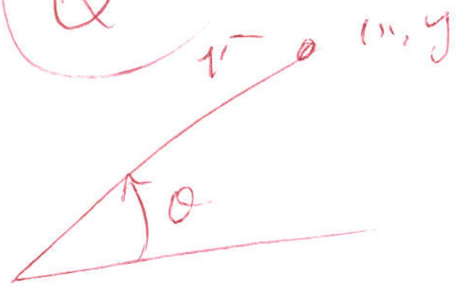
$= 0$ $\begin{cases} \text{Because of the equality of} \\ \text{mixed partial derivatives.} \end{cases}$

(v) $\nabla \cdot (\nabla \phi) = \nabla \cdot \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Q9



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r}$$

$$\Rightarrow \boxed{\phi_r = \phi_x \cos\theta + \phi_y \sin\theta}$$

$$\boxed{\xi_r = \xi_x \cos\theta + \xi_y \sin\theta}$$

so

$$\Rightarrow \phi_{rr} = \phi_{xr}\cos\theta + \phi_{yr}\sin\theta$$

$$= [\phi_x]_r \cos\theta + [\phi_y]_r \sin\theta$$

$$= \underbrace{\left[\phi_{xx}\cos\theta + \phi_{xy}\sin\theta\right]}_{\xi = \phi_x}\cos\theta + \underbrace{\left[\phi_{yx}\cos\theta + \phi_{yy}\sin\theta\right]}_{\xi = \phi_y}\sin\theta$$

$$\Rightarrow \boxed{\phi_{rr} = \phi_{xx}\cos^2\theta + 2\phi_{xy}\sin\theta\cos\theta + \phi_{yy}\sin^2\theta}$$

$$\phi_\theta = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta}$$

$$\Rightarrow \boxed{\phi_\theta = \phi_x(-r\sin\theta) + \phi_y(r\cos\theta)}$$

so $\boxed{\xi_\theta = r(\xi_y\cos\theta - \xi_x\sin\theta)}$

$$\phi_{\theta\theta} = [\phi_x]_\theta(-r\sin\theta) + \phi_x(-r\cos\theta)$$

$$+ [\phi_y]_\theta(r\cos\theta) + \phi_y(-r\sin\theta)$$

$$= [r(\phi_{xy}\cos\theta - \phi_{xx}\sin\theta)](-r\sin\theta)$$

Put $\xi = \phi_x$

$$+ [r(\phi_{yy}\cos\theta - \phi_{yx}\sin\theta)](r\cos\theta)$$

Put $\xi = \phi_y$

$$- r(\phi_x\cos\theta + \phi_y\sin\theta)$$

3

$$\Rightarrow \phi_{\theta\theta} = r^2 \left[\phi_{yy} \cos^2\theta + \phi_{xx} \sin^2\theta - 2\phi_{xy} \sin\theta\cos\theta \right] - r\left[\phi_r\right]$$

$$\Rightarrow \frac{1}{r^2}\phi_{\theta\theta} + \frac{1}{r}\phi_r = \phi_{yy}\cos^2\theta - 2\phi_{xy}\sin\theta\cos\theta + \phi_{xx}\sin^2\theta$$

$$\Rightarrow \boxed{\phi_{rr} + \frac{1}{r^2}\phi_{\theta\theta} + \frac{1}{r}\phi_r}$$

$$= \phi_{xx}\cos^2\theta + \cancel{2\phi_{xy}\sin\theta\cos\theta} + \phi_{yy}\sin^2\theta$$

$$+ \phi_{xx}\sin^2\theta - \cancel{2\phi_{xy}\sin\theta\cos\theta} + \phi_{yy}\cos^2\theta$$

$$= \phi_{xx}\underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{=1} + \phi_{yy}\underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{=1}$$

$$= \phi_{xx} + \phi_{yy} = \nabla^2\phi$$