MS 221 HOMEWORK SET (8)

$\boxed{Q1}$ $\gamma : [0,1] \longrightarrow \mathbb{R}^3 : t \longmapsto \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

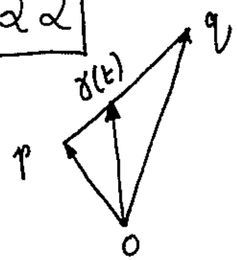
parametrizes the curve $\mathscr{C}$.

$$\int_{\mathscr{C}} xy\,dz = \int_0^1 x(t)\,y(t)\,\frac{dz}{dt}(t)\,dt$$

$$= \int_0^1 1.(2t).(6t)\,dt$$

$$= \int_0^1 12t^2\,dt = \left[4t^3\right]_0^1 = 4.$$

$\boxed{Q2}$



$\gamma : [0,1] \longrightarrow \mathbb{R}^3 : t \longmapsto p + t(q-p)$

parametrizes the line segment from p to q. Thus

$$\gamma(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1-1 \\ 2-0 \\ 3-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \\ 3t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\int_{\mathscr{C}} xy\,dz = \int_0^1 x(t)\,y(t)\,\dot{z}(t)\,dt = \int_0^1 1.(2t).3\,dt$$

$$= \left[3t^2\right]_0^1 = 3.$$

$\boxed{Q3}$ $\gamma(t) = \begin{bmatrix} 2t \\ 1-t \\ 2+t^2 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad \forall\, t \in [1,3].$

$$\int_{\mathscr{C}} 3x\,dx - y^2\,dy + dz = \int_1^3 \left[3x(t)\,\dot{x}(t) - y^2(t)\,\dot{y}(t) + \dot{z}(t)\right]dt$$

$$= \int_1^3 \left[3(2t)2 - (1-t)^2.(-1) + 2t\right]dt$$

$$= \int_1^3 \left[14t + (t-1)^2\right]dt$$

$$= \left[7t^2 + \frac{(t-1)^3}{3}\right]_1^3 = 58\tfrac{2}{3}.$$

$\boxed{Q4}$ $\text{Work} = \int_{-1}^1 \left\langle F(\gamma(t)),\ \frac{d\gamma}{dt} \right\rangle dt$

$$= \int_{-1}^1 \left\langle \begin{bmatrix} t^2-(2-t)^2 \\ -4 \\ t^2(2-t)16 \end{bmatrix}, \begin{bmatrix} 2t \\ -1 \\ 0 \end{bmatrix} \right\rangle dt$$

$$= \int_{-1}^1 \left[(4t-4).2t + 4\right]dt = \frac{40}{3}.$$

Q5 Here the curve

$$\gamma(x) = \begin{bmatrix} x \\ \sqrt{9-x^2} \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y(x) \\ z(x) \end{bmatrix} \quad \forall x \in [-3,3]$$

is the semi-circle $y = \sqrt{9-x^2}$ which lies on the plane $z = 2$. That is, $x^2+y^2 = 9$ (with $y \geq 0$) and $z = 2$. This we can parametrize by

$$\gamma : [-\pi, 0] \longrightarrow \mathbb{R}^3 : t \longmapsto \begin{bmatrix} 3\cos t \\ -3\sin t \\ 2 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\text{Work} = \int_{-\pi}^{0} \left\langle F(\gamma(t)), \frac{d\gamma}{dt}(t) \right\rangle dt$$

$$= \int_{-\pi}^{0} \left\langle \begin{bmatrix} -9\cos t \sin t \\ 6\sin t \\ 6\cos t \end{bmatrix}, \begin{bmatrix} -3\sin t \\ -3\cos t \\ 0 \end{bmatrix} \right\rangle dt$$

$$= \int_{-\pi}^{0} 9\left[3\cos t \sin^2 t - 2\sin t \cos t \right] dt$$

$$= 9\left[\sin^3 t - \sin^2 t \right]_{-\pi}^{0} = 0$$

Q6 $\text{Work} = \int_0^1 \left\langle F(\gamma(t)), \frac{d\gamma}{dt}(t) \right\rangle dt$

since $F = \nabla\varphi$

$$= \int_0^1 \left\langle \nabla\varphi_{\gamma(t)}, \frac{d\gamma}{dt}(t) \right\rangle dt$$

By The Chain Rule

$$= \int_0^1 \frac{d}{dt}\varphi(\gamma(t))\, dt$$

$$= \varphi(\gamma(t)) \Big|_{t=0}^{t=1}$$

$$= \varphi(\gamma(1)) - \varphi(\gamma(0))$$

$$= \varphi(-1,1,1) - \varphi(1,1,0)$$

$\varphi(x,y,z) = x(y^2+z^2)$

$$= -1(1+1) - 1(1+0)$$

$$= -3.$$

Q7

$$\int_0^1 \int_{y^2}^{y} 2xy \, dx \, dy = \int_0^1 \left[x^2 y\right]_{x=y^2}^{x=y} dy$$

$$= \int_0^1 \left[y^3 - y^5\right] dy$$

$$= \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

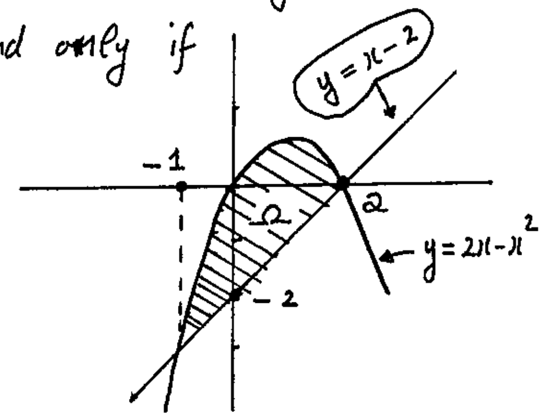
Q8 The curves $y = x - 2$ and $y = 2x - x^2$ intersect if and only if

$$x - 2 = 2x - x^2$$

$$\iff x^2 - x - 2 = 0$$

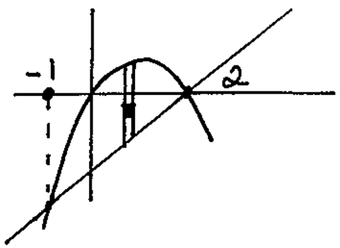
$$\iff (x+1)(x-2) = 0$$

$$\iff x = -1 \text{ or } 2.$$



$$\iint_\Omega f(x,y)\, dA$$

$$= \int_{-1}^{2} \int_{y=x-2}^{y=2x-x^2} f(x,y)\, dy\, dx$$

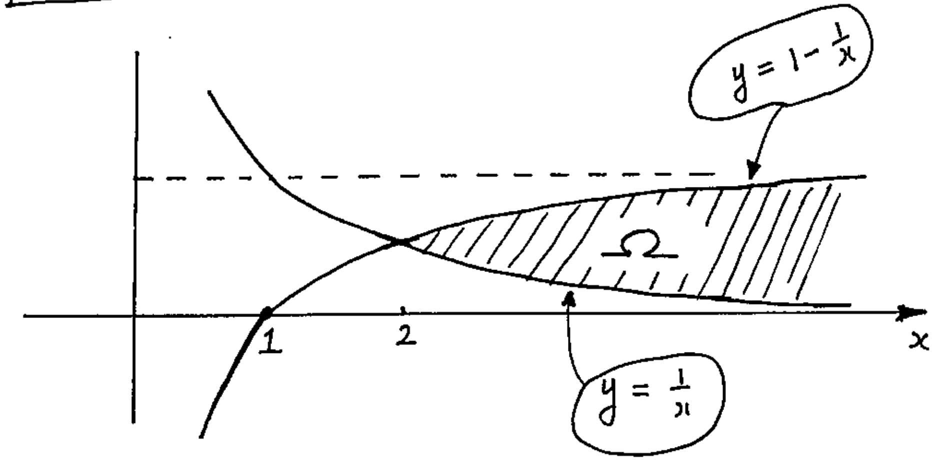


Q9 When $x > 0$, the curves

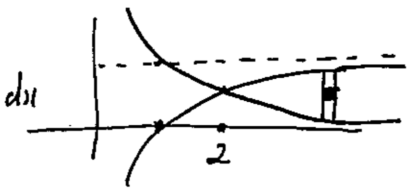
$$y = \frac{1}{x} \quad \text{and} \quad y = 1 - \frac{1}{x}$$

intersect if and only if

$$\boxed{\frac{1}{x} = 1 - \frac{1}{x}} \iff \boxed{\frac{2}{x} = 1} \iff \boxed{x = 2}.$$



$$\iint_\Omega \frac{2y}{x^2}\, dA = \int_2^\infty \int_{y=\frac{1}{x}}^{y=1-\frac{1}{x}} \frac{2y}{x^2}\, dy\, dx$$



$$= \int_2^\infty \frac{1}{x^2}\left[y^2\right]_{y=\frac{1}{x}}^{y=1-\frac{1}{x}} dx$$

$$= \int_2^\infty \frac{1}{x^2}\left[\left(1 - \frac{1}{x}\right)^2 - \frac{1}{x^2}\right] dx = \int_2^\infty \left[\frac{1}{x^2} - \frac{2}{x^3}\right] dx$$

$$= \left[-\frac{1}{x} + \frac{1}{x^2}\right]_{x=2}^{x=\infty} = 0 - \left[-\frac{1}{2} + \frac{1}{4}\right] = \frac{1}{4}.$$