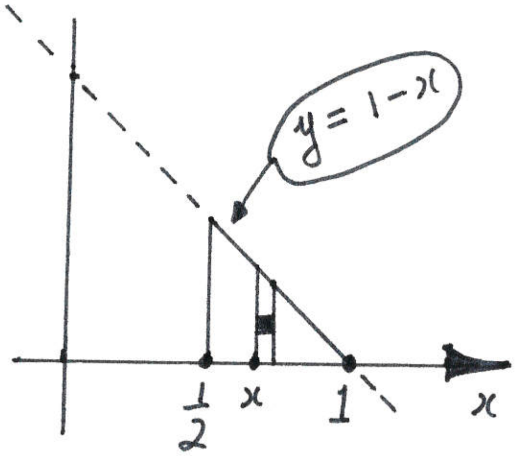
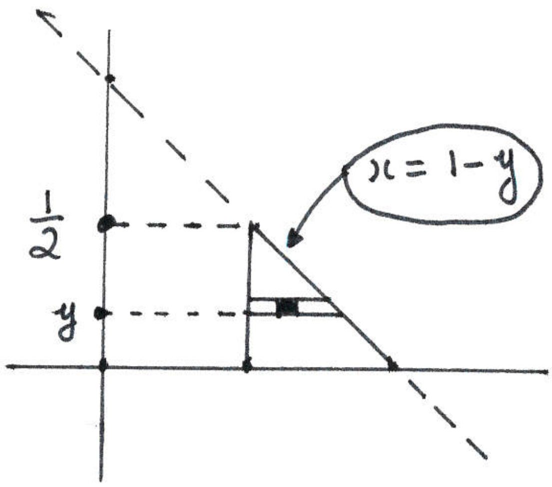
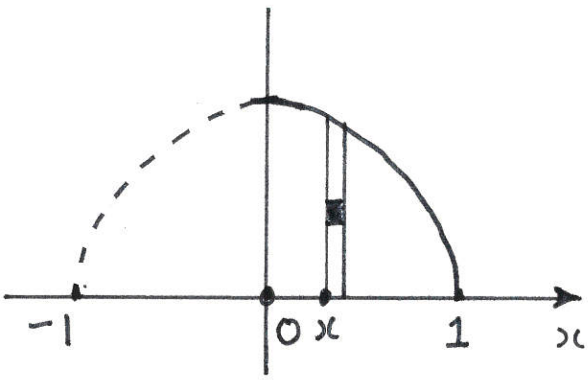
MS 221 HOMEWORK SET (9)

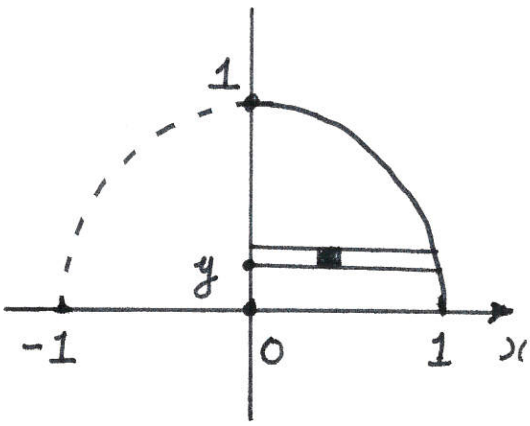
(i) $$\int_{1/2}^{1}\int_{0}^{1-x} f(x,y)\,dy\,dx = \int_{0}^{1/2}\int_{x=1/2}^{x=1-y} f(x,y)\,dx\,dy$$



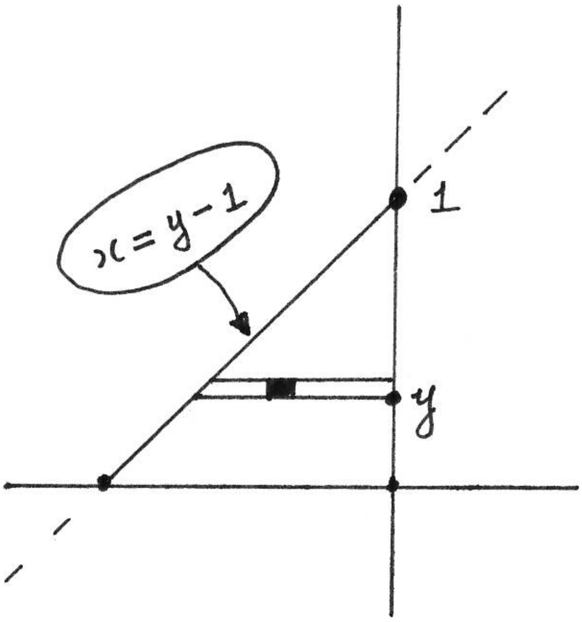


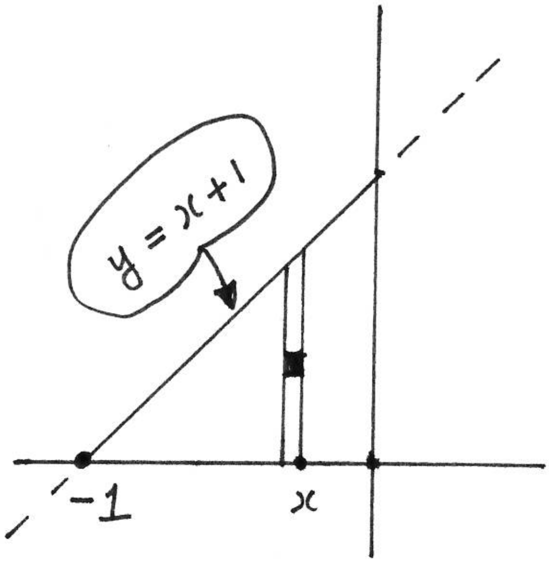
(ii) $$\int_{0}^{1}\int_{y=0}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{0}^{1}\int_{x=0}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$



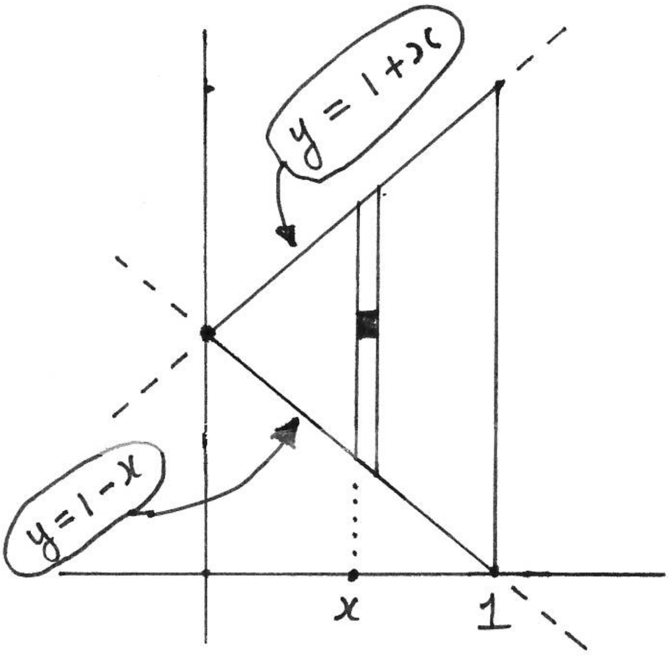


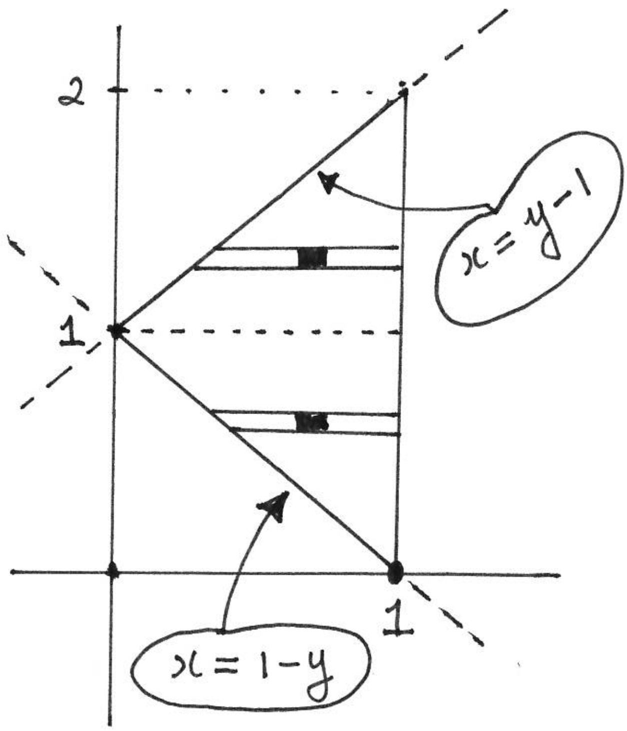
(iii) $$\int_0^1 \int_{y-1}^{0} f(x,y)\,dx\,dy = \int_{-1}^{0} \int_{y=0}^{y=x+1} f(x,y)\,dy\,dx$$





(iv) $$\int_0^1 \int_{1-x}^{1+x} f(x,y)\,dy\,dx = \int_0^1 \int_{1-y}^{1} f(x,y)\,dx\,dy + \int_1^2 \int_{y-1}^{1} f(x,y)\,dx\,dy$$





Q2

$$\int_0^1 \int_{x-1}^0 \int_0^{1-x+y} x \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x-1}^0 \left[xz \right]_{z=0}^{z=1-x+y} dy \, dx$$

$$= \int_0^1 \int_{x-1}^0 \left[x - x^2 + xy \right] dy \, dx$$

$$= \int_0^1 \left[xy - x^2 y + \frac{xy^2}{2} \right]_{y=x-1}^{0} dx$$

$$= \int_0^1 - \left[x(x-1) - x^2(x-1) + \frac{x(x-1)^2}{2} \right] dx$$

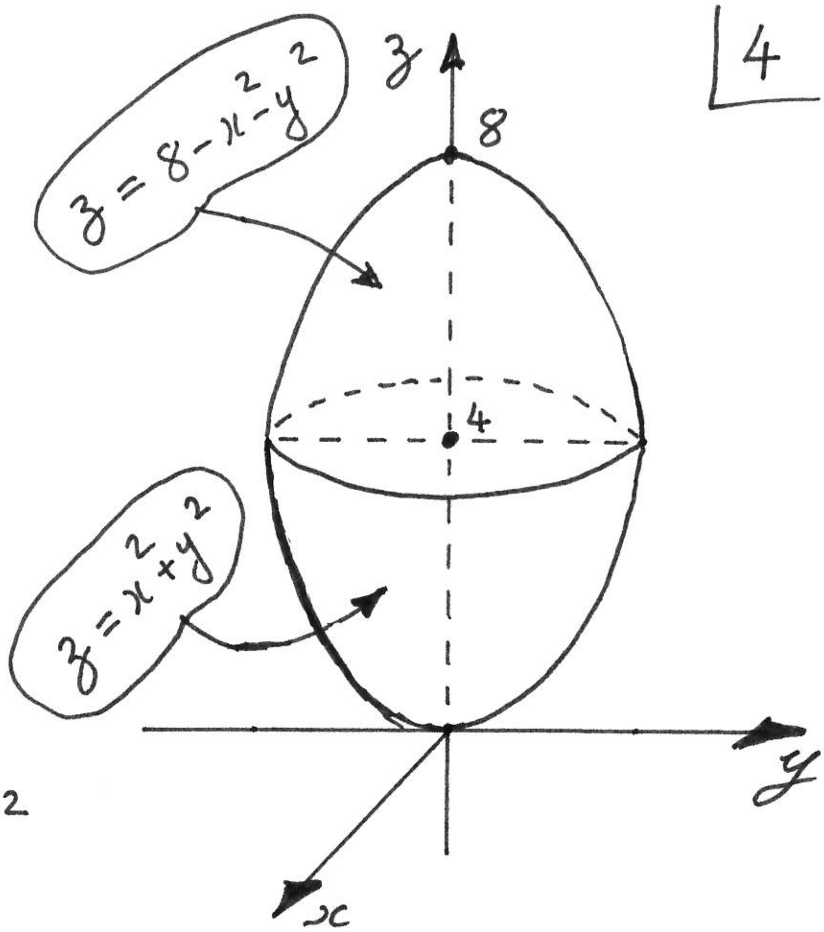
$$= \frac{1}{2} \int_0^1 \left[x^3 - 2x^2 + x \right] dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right]$$

$$= 1/24.$$

Q3



The surfaces

$$z = x^2 + y^2$$

and

$$z = 8 - x^2 - y^2$$

intersect when

$$x^2 + y^2 = z = 8 - x^2 - y^2$$

$$\Leftrightarrow \quad 2(x^2 + y^2) = 8$$

$\Leftrightarrow \quad x^2 + y^2 = 4$. That is, a circle of radius $= 2$ on the plane $z = 4$.

$\mathcal{V}$ is the volume inside the two surfaces above. Thus

$$\iiint_{\mathcal{V}} (x^2+y^2)\, dV = \iint_{x^2+y^2 \leq 4} \int_{z = x^2+y^2}^{z = 8-(x^2+y^2)} (x^2+y^2)\, dz\, dA$$

$$= \iint_{x^2+y^2 \leq 4} \Big[(x^2+y^2)\, z \Big]_{z = x^2+y^2}^{z = 8-(x^2+y^2)} dA.$$

$$= 2 \iint_{x^2+y^2 \leq 4} \left[4(x^2+y^2) - (x^2+y^2)^2 \right] dA$$

Change to polar coords.

$$\overset{\downarrow}{=} 2 \int_{0}^{2\pi} \int_{r=0}^{r=2} \left[4r^2 - r^4 \right] r \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} \int_{r=0}^{r=2} \left[4r^3 - r^5 \right] dr \, d\theta$$

$$= 2 \int_0^{2\pi} \left[r^4 - \frac{r^6}{6} \right]_{r=0}^{r=2} d\theta$$

$$= 2 \int_0^{2\pi} \left[16 - \frac{32}{3} \right] d\theta$$

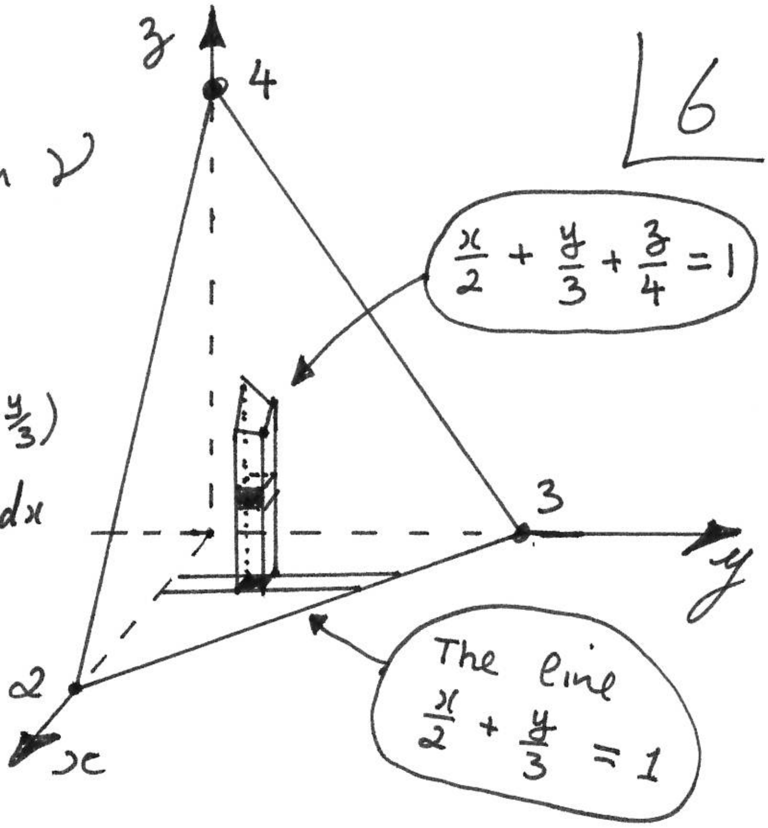
$$= \frac{32}{3} \, \theta \Big|_0^{2\pi}$$

$$= \frac{64\pi}{3}.$$

$\boxed{Q4}$ Total Charge in $\mathcal{V}$

$$= \iiint_{\mathcal{V}} \rho(x,y,z)\, dV$$

$$= \int_0^2 \int_{y=0}^{y=\frac{3}{2}(2-x)} \int_{z=0}^{z=4\left(1-\frac{x}{2}-\frac{y}{3}\right)} (x+y+z)\, dz\, dy\, dx$$



$$= \int_0^2 \int_{y=0}^{y=\frac{3}{2}(2-x)} \left[xz + yz + \frac{z^2}{2} \right]_{z=0}^{z=4\left(1-\frac{x}{2}-y/3\right)} dy\, dx$$

$$= \int_0^2 \int_{y=0}^{y=\frac{3}{2}(2-x)} \left[8 - 4x - \frac{4}{3}y - \frac{2}{3}xy - \frac{4}{9}y^2 \right] dy\, dx$$

$$= \int_0^2 \left[14 - 15x + \frac{9}{2}x^2 - \frac{x^3}{4} \right] dx$$

$$= 9.$$

Q5 The surface $z = x^2 + y^2$ and $z = 2x$ intersect when

$$x^2 + y^2 = z = 2x$$

$$\Longleftrightarrow \quad x^2 - 2x + y^2 = 0 \quad \text{and} \quad z = 2x$$

$$\Longleftrightarrow \quad (x-1)^2 + y^2 = 1 \quad \text{and} \quad z = 2x$$

Thus the volume of the region in question

$$= \iint_{(x-1)^2+y^2 \leq 1} \int_{z=x^2+y^2}^{z=2x} dz \, dA$$

$$= \iint_{(x-1)^2+y^2 \leq 1} \Big[z \Big]_{z=x^2+y^2}^{z=2x} dA$$

$$= \iint_{(x-1)^2+y^2 \leq 1} \left[1 - (x-1)^2 - y^2 \right] dA$$

Now change to polar coordinates centred at $(1,0)$. Thus $r^2 = (x-1)^2 + y^2$.

$$= \int_0^{2\pi} \int_{r=0}^{r=1} \left[1 - r^2 \right] r \, dr \, d\theta = \frac{\pi}{2}.$$

$\boxed{Q6}$

$$x(u,v) = u^2 + 2uv$$
$$y(u,v) = v^2 + 2uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2(u+v) & 2u \\ 2v & 2(u+v) \end{bmatrix}$$

$$dA = \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \left| 4(u+v)^2 - 4uv \right| du\,dv$$

$$= 4 \left| u^2 + uv + v^2 \right| du\,dv$$

Q7 $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

$$= \left(e^x \cos y\right)^2 + \left(e^x \sin y\right)^2$$

$$= u^2 + v^2$$

$$dA = \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \left| \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}} \right| du\,dv$$

$$= \frac{1}{u^2+v^2}\, du\,dv.$$

Q8

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$dV = \left| \det \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| dr\, d\theta\, dz$$

$$= \left| \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} \right| dr\, d\theta\, dz$$

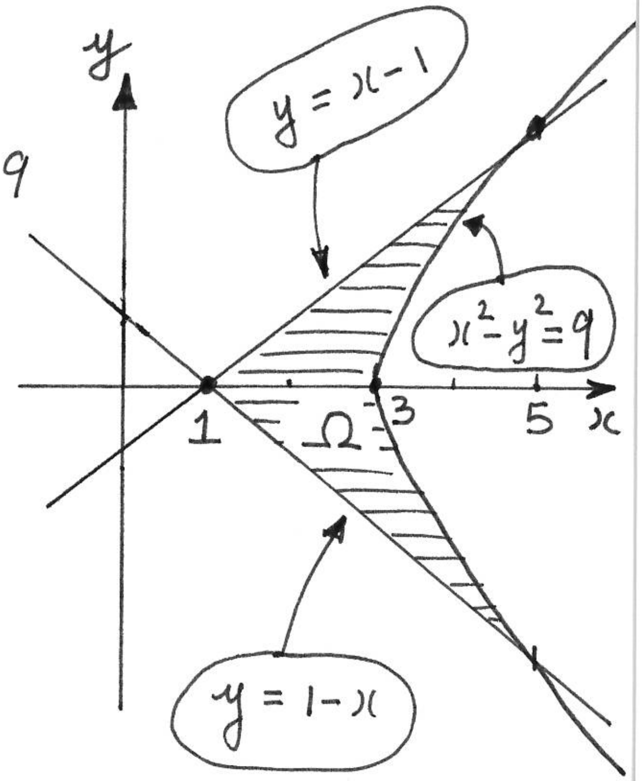
$$= \left| \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| dr\, d\theta\, dz$$

$$= r\,(\cos^2\theta + \sin^2\theta)\, dr\, d\theta\, dz$$

$$= r\, dr\, d\theta\, dz.$$

Q9 The curve $x^2 - y^2 = 9$ intersects the lines $y = \pm(x-1)$

$$\iff x^2 - (x-1)^2 = 9$$
$$\iff x^2 - (x^2 - 2x + 1) = 9$$
$$\iff 2x - 1 = 9$$
$$\iff x = 5$$



The transformation

$$(x, y) \longmapsto (u, v)$$

is given by

$$\boxed{\begin{aligned} u &= x + y \\ v &= x^2 - y^2 \end{aligned}} \longleftrightarrow \boxed{\begin{aligned} (x+y) &= u \\ (x+y)(x-y) &= v \end{aligned}}$$

$$\updownarrow$$

$$\boxed{\begin{aligned} x+y &= u \\ x-y &= \frac{v}{u} \end{aligned}}$$

Accordingly, we obtain the correspondences

$$\boxed{y = x - 1} \longleftrightarrow \boxed{x - y = 1} \longleftrightarrow \boxed{v = u}$$

$$\boxed{y = 1 - x} \longleftrightarrow \boxed{x + y = 1} \longleftrightarrow \boxed{u = 1}$$

$$\boxed{x^2 - y^2 = 9} \longleftrightarrow \boxed{v = 9}$$

These are the boundary curves to Ω in the (x,y) plane

These are the corresponding boundary curves to the image of Ω in the (u,v)-plane