MS 221 — Homework Set (10)

(Surface Integrals /Taylors Theorem)

QUESTION 1

Use the **change of variables**

$$u = x + y, \qquad v = \frac{y}{x+y}$$

to show that

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dydx = \frac{e-1}{2}$$

QUESTION 2

Let Ω be some fixed region in the xy −plane and let $\boldsymbol{\mathcal{S}} \subset \boldsymbol{R}^3$ be the surface given by

$$z = (x-y)^2 \quad \forall \ (x,\, y) \in \Omega.$$

Denote the **upward pointing** unit normal field to this surface by $\boldsymbol{n}$. If the **vector field** $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is defined by

$$\boldsymbol{F}(x,\, y,\, z) = \begin{bmatrix} x+y \\ 0 \\ 2z \end{bmatrix}.$$

determine the function $f : \Omega \to \boldsymbol{R} : (x,\, y) \mapsto f(x,\, y)$ such that

$$\int\int_{\boldsymbol{\mathcal{S}}} \langle \boldsymbol{F},\, \boldsymbol{n} \rangle \, dA_{\boldsymbol{\mathcal{S}}} \ = \ \int\int_{\Omega} f(x,\, y)\, dx\, dy.$$

QUESTION 3

Show that the vector field

$$F : \boldsymbol{R}^3 \to \boldsymbol{R}^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \sin y \\ x\cos y + \sin z \\ y \cos z \end{bmatrix}$$

is conservative and find a **scalar potential** φ.

QUESTION 4

Throughout this question Ω will denote the disc in the xy-plane which is centred at the origin and has radius 2, that is $\Omega = \{(x,\, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 \leq 4\}$ and $\boldsymbol{\mathcal{S}}$ will denote the surface in $\boldsymbol{R}^3$ given by

$$z = 4 - (x^2 + y^2) \qquad \forall\ (x,\, y) \in \Omega$$

If $\boldsymbol{F}$ is the vector field

$$\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + yz \\ y + xz \\ xyz \end{bmatrix}$$

do the following:

(a) Sketch the surface $\boldsymbol{\mathcal{S}}$ together with its boundary curve $\boldsymbol{\mathcal{C}}$.

(b) Calculate the unit (upward pointing) normal field to the surface $\boldsymbol{\mathcal{S}}$.

(c) Determine the function $\varphi : \Omega \to \boldsymbol{R} : (x,\, y) \mapsto \varphi(x,\, y)$ such that

$$\iint_{\boldsymbol{\mathcal{S}}} \langle \boldsymbol{\nabla} \times \boldsymbol{F},\, \boldsymbol{n} \rangle \, dA_{\boldsymbol{\mathcal{S}}} \;=\; \iint_{\Omega} \varphi(x,\, y)\, dxdy.$$

Note: You are **NOT** asked to evaluate this integral.

(d) Using Stokes' Theorem, or otherwise, evaluate

$$\iint_{\boldsymbol{\mathcal{S}}} \langle \boldsymbol{\nabla} \times \boldsymbol{F},\, \boldsymbol{n} \rangle \, dA_{\boldsymbol{\mathcal{S}}}$$

where $\boldsymbol{n}$ is the normal field obtained in part (b).

QUESTION 5

Find the **Taylor series** of the function $f(x,\, y) = x^3 - y^2 + y$ about the point $(2,\, -3)$.

QUESTION 6

Find all terms **up to second order** in the **Taylor series** of the function $f(x,\, y) = \sin(xy)$ about the point $(0,\, -1)$.