MS 221 — Homework Set (5)

(Applications of The Chain Rule)

QUESTION 1

A particle moving on a plane has Cartesian and polar coordinates at time t given by $(x(t),\, y(t))$ and $(r(t),\, \theta(t))$, respectively. Thus,

$$x(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad y(t) = r(t)\sin\theta(t) \quad \forall\ t \in \boldsymbol{R}.$$

If the speed in Cartesian coordinates is given by $\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$ find the corresponding formula for the speed in terms of polar coordinates, that is, in terms of r, θ, $\dot{r}$ and $\dot{\theta}$.

QUESTION 2

A disc with **centre at the origin** rotates anti-clockwise with **constant angular speed** ω revolutions/sec about the origin. An insect on this disc is crawling in a straight line (relative to the disc) towards the centre at a constant speed (relative to the disc) of α cm/sec. If the polar coordinates of the insect at time $t = 0$ are $r(0) = 100$ cm and $\theta(0) = 0$ radians do the following:

(a) Find the polar coordinates $(r(t),\, \theta(t))$ of the insect at any subsequent time t.

(b) Use part (a) to determine the Cartesian coordinates $(x(t),\, y(t))$ of the insect at any subsequent time t.

(c) Find the velocity and acceleration (vectors) in Cartesian coordinates of the insect at any subsequent time t.

QUESTION 3

A function $f : \boldsymbol{R}^2 \to \boldsymbol{R} : (x,\, y) \mapsto f(x,\, y)$ satisfies

$$\frac{\partial f}{\partial x}(0,\, 0) = 3 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,\, 0) = -5.$$

If in addition, $f(ta,\, tb) = tf(a,\, b)$ for every $t \in \boldsymbol{R}$ and for every $(a,\, b) \in \boldsymbol{R}^2$, find $f(a,\, b)\ \forall\ (a,\, b) \in \boldsymbol{R}^2$.

Hint: $tf(a,\, b) \equiv f(ta,\, tb) \implies f(a,\, b) \equiv \dfrac{d}{dt} f(ta,\, tb).$

QUESTION 4

Express the partial derivative $\dfrac{\partial}{\partial x} f(\, u(x,\, y),\, v(x,\, y),\, w(x,\, y)\,)$ in terms of the **Chain Rule**.

QUESTION 5

Throughout this question Ω will denote the set in the xy-plane given by:

$$\Omega = \{ (x,\, y) \in \boldsymbol{R}^2 \mid y > 0 \}.$$

If the functions $\xi : \Omega \to \boldsymbol{R}$ and $\eta : \Omega \to \boldsymbol{R}$ are specified by

$$\xi(x,\, y) = x \ln y \quad \text{and} \quad \eta(x,\, y) = x,$$

express the partial differential equation

$$x\frac{\partial u}{\partial x} - y \ln y \, \frac{\partial u}{\partial y} = u \qquad \text{on } \Omega$$

as a partial differential equation in the (ξ, η) - coordinates and, hence or otherwise, solve this partial differential equation subject to the condition that

$$u(x, e) \equiv xe^x \quad \text{for all } x \in \boldsymbol{R}$$

QUESTION 6

Notation: In the case where $\omega = f(\, u(x,\, t),\, v(x,\, t)\,)$ we will write $\dfrac{\partial \omega}{\partial u} := \dfrac{\partial f}{\partial u}(u,\, v)$,

$$\frac{\partial \omega}{\partial v} := \frac{\partial f}{\partial v}(u,\, v), \qquad \frac{\partial \omega}{\partial x} := \frac{\partial f}{\partial x}(\, u(x,\, t),\, v(x,\, t)\,), \qquad \frac{\partial \omega}{\partial t} := \frac{\partial f}{\partial t}(\, u(x,\, t),\, v(x,\, t)\,)$$

and similarly for higher order derivatives. Now, if

$$u(x,\, t) = x + ct \quad \text{and} \quad v(x,\, t) = x - ct,$$

where c is a non-zero constant, show that

$$\frac{\partial^2 \omega}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \omega}{\partial t^2} \equiv 4\frac{\partial^2\, \omega}{\partial u \partial v}$$

QUESTION 7

Find all solutions of the (partial differential) equation $\dfrac{\partial^2\, \omega}{\partial u \partial v} \equiv 0.$

Hint: If a function $h(r,\, s)$ satisfies $\dfrac{\partial h}{\partial r} \equiv 0$, then h is constant in r. That is, h **is a function of** s **only**.

QUESTION 8

Use Questions 6 and 7 above to show that **every solution** ω of the 1-dimensional wave equation

$$\frac{\partial^2 \omega}{\partial x^2} \equiv \frac{1}{c^2}\frac{\partial^2 \omega}{\partial t^2}$$

is of the form $\omega = \varphi(x + ct) + \psi(x - ct)$ where φ and ψ are arbitrary smooth functions