MS 221 — Homework Set (6)

(The Chain Rule and The Gradient)

QUESTION 1

Let $\boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and let the function $f : \boldsymbol{R}^3 \to \boldsymbol{R} : \begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto f(u, v, w)$ satisfy

$$\frac{\partial f}{\partial u}(\boldsymbol{p}) = -3, \quad \frac{\partial f}{\partial v}(\boldsymbol{p}) = 2, \quad \frac{\partial f}{\partial w}(\boldsymbol{p}) = 5.$$

If the functions $u, v, w : \boldsymbol{R}^2 \to \boldsymbol{R}$ are defined by

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= x + xy \\ w(x, y) &= 2 + xy - y^3 \end{aligned}$$

calculate

$$\frac{\partial}{\partial x} f(u(x, y), v(x, y), w(x, y)) \quad \text{at} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

QUESTION 2

Find $\nabla f_{\boldsymbol{p}}$ where the function $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$f(x,\, y,\, z) = x^2 z + y \ln(z^2 + 1) \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

QUESTION 3

Find all points $\boldsymbol{p} \in \boldsymbol{R}^3$ such that $\nabla \varphi_{\boldsymbol{p}} = \boldsymbol{0}$ where the function $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ is given by

$$\varphi(x,\, y,\, z) = x^2 y + y^2 z + z^2 x.$$

QUESTION 4

Let $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ be as in Question 3 and let M be the **level set** determined by

$$\varphi(x,\, y,\, z) = 1.$$

Now, show that the point $\boldsymbol{p} = (1,\, -1,\, 1)$ is on the level set M and find the equation of the **tangent plane** to M at $\boldsymbol{p}$.

QUESTION 5

The equation $z = f(x,\, y)$ defines a surface M in $\boldsymbol{R}^3$. If we put $z_0 = f(x_0,\, y_0)$, then the point $\boldsymbol{p}_0 = (x_0,\, y_0,\, z_0)$ is on the surface M. Now, find the equation of the **tangent plane** to M at $\boldsymbol{p}_0$.

Hint: Define the function $\varphi(x,\, y,\, z) \equiv z - f(x,\, y)$, then

$$\boxed{z = f(x,\, y)} \quad \Longleftrightarrow \quad \boxed{\varphi(x,\, y,\, z) = 0}$$

QUESTION 6

Let the functions $\varphi,\, \psi : \boldsymbol{R}^2 \to \boldsymbol{R}$ be given by:

$$\varphi(x,\, y) = x^2 - y^2 + x \qquad \text{and} \qquad \psi(x,\, y) = 2xy + y$$

If $\boldsymbol{\mathcal{C}}_1,\, \boldsymbol{\mathcal{C}}_2 \subset \boldsymbol{R}^2$ denote the **level sets** (i.e. curves) defined by

$$\varphi(x,\, y) \equiv 0 \qquad \text{and} \qquad \psi(x,\, y) \equiv 0$$

respectively, show that $(0,\, 0) \in \boldsymbol{\mathcal{C}}_1 \cap \boldsymbol{\mathcal{C}}_2$ and find the **angle of intersection** of $\boldsymbol{\mathcal{C}}_1$ and $\boldsymbol{\mathcal{C}}_2$ at this point.