MS 221 — Homework Set (7)

(Lagrange Multipliers / Grad, Div and Curl)

QUESTION 1

Determine the **shortest distance** from the point $(0, b)$ on the y-axis to the parabola $x^2 - 4y = 0$ in each of the following ways:

(i) Use the method of **Lagrange multipliers**.

(ii) Use the constraint $x^2 - 4y = 0$ to **eliminate one of the variables**, thus reducing the problem to the calculus of one variable.

Hint: Distance is minimized $\Longleftrightarrow$ $(\text{Distance})^2$ is minimized

QUESTION 2

Let $\wp$ be the plane in $\boldsymbol{R}^3$ which passes through the point $\boldsymbol{p}$ and is normal to the vector $\boldsymbol{n}$. If $\boldsymbol{q}$ is any point in $\boldsymbol{R}^3$, use the method of **Lagrange multipliers** to find the shortest distance from the point $\boldsymbol{q}$ to the plane $\wp$.

QUESTION 3

The cone $z^2 = x^2 + y^2$ is cut by the plane $2x + 2y + 2z = 4$ in a curve $\boldsymbol{\mathcal{C}}$. Find the points on $\boldsymbol{\mathcal{C}}$ which are nearest and furthest away from the xy-plane.

QUESTION 4

Use the method of **Lagrange multipliers** to find the points on the curve

$$3x^2 - 8xy - 3y^2 = 5$$

which are nearest and furthest away from the origin.

QUESTION 5

Calculate $\nabla\varphi_{\boldsymbol{p}}$ (that is, the **gradient** of φ at $\boldsymbol{p}$) where the function $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$\varphi(x,\, y,\, z) = x^2 z + e^{yz} \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{respectively.}$$

QUESTION 6

Calculate $\nabla .\, \boldsymbol{F}_{\boldsymbol{p}}$, (that is, the **divergence** of $\boldsymbol{F}$ at $\boldsymbol{p}$) where the vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are given by

$$\boldsymbol{F}(x,\, y,\, z) = \begin{bmatrix} x^2 y \\ x - yz \\ \sin(yz) \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \qquad \text{respectively.}$$

QUESTION 7

Calculate $\nabla \times \boldsymbol{F}_{\boldsymbol{p}}$, (that is, the **curl** of $\boldsymbol{F}$ at $\boldsymbol{p}$) where the vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and the point $\boldsymbol{p} \in \boldsymbol{R}^3$ are as given in Question 4

QUESTION 8

In the case of any (smooth) scalar field $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ and vector field $\boldsymbol{F} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ establish the following

(i) $\nabla .\, (\varphi\, \boldsymbol{F}) = (\nabla\varphi) \,.\, \boldsymbol{F} + \varphi\, (\nabla .\, \boldsymbol{F})$.

(ii) $\nabla \times (\varphi\, \boldsymbol{F}) = (\nabla\varphi) \times \boldsymbol{F} + \varphi\, (\nabla \times \boldsymbol{F})$.

(iii) $\nabla \times (\nabla\varphi) \equiv \boldsymbol{0}$.

(iv) $\nabla .\, (\nabla \times \boldsymbol{F}) \equiv 0$.

(v) $\nabla .\, (\nabla\varphi) = \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial z^2}$.

QUESTION 9

Use the **Chain Rule** to express the two dimensional **Laplacian**

$$\nabla .\, (\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

in terms of **polar coordinates**.