MS 221 — Homework Set (9)

(Multiple Integration & Change of Variables)

QUESTION 1

In the case of each of the following iterated integrals write the corresponding iterated integral with the order of integration reversed:

(i) $\displaystyle\int_{1/2}^{1}\int_{0}^{1-x} f(x,y)\,dydx$ (ii) $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dydx$

(iii) $\displaystyle\int_{0}^{1}\int_{y-1}^{0} f(x,y)\,dxdy$ (iv) $\displaystyle\int_{0}^{1}\int_{1+x}^{1-x} f(x,y)\,dydx$

QUESTION 2

Evaluate the iterated integral $\displaystyle\int_{0}^{1}\int_{x-1}^{0}\int_{0}^{1-x+y} x\,dz\,dy\,dx.$

QUESTION 3

Calculate the volume integral $\displaystyle\iiint_{\boldsymbol{\mathcal{V}}} (x^2+y^2)\,dV$ where the volume $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{R}^3$ is determined by the inequalities:

$$x^2+y^2 \;\le\; z \;\le\; 8-x^2-y^2.$$

QUESTION 4

Let the volume $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{R}^3$ be determined by the inequalities:

$$0\le x,\quad 0\le y,\quad 0\le z,\quad \text{and}\quad \frac{x}{2}+\frac{y}{3}+\frac{z}{4}\le 1.$$

If the **charge density**, per unit volume, inside this region is given by $\rho(x,\,y,\,z) = x+y+z$ calculate the total charge in $\boldsymbol{\mathcal{V}}$.

QUESTION 5

Find the volume of the region in $\boldsymbol{R}^3$ which is determined by the inequalities:

$$x^2+y^2\le z\le 2x.$$

QUESTION 6

If the rectangular Cartesian coordinates (x, y) and the curvilinear coordinates (u, v) are related by

$$x = u^2 + 2uv \qquad \text{and} \qquad y = 2uv + v^2$$

calculate the **Jacobian matrix**

$$\frac{\partial(x, y)}{\partial(u, v)}$$

and hence find the function $\varphi(u, v)$ such that the **area element** dA in terms of the (u, v) coordinates is given by $dA = \varphi(u, v)\, dudv$.

QUESTION 7

If the rectangular Cartesian coordinates (x, y) and the curvilinear coordinates (u, v) are related by

$$u = e^x \cos y \qquad \text{and} \qquad v = e^x \sin y$$

express the **Jacobian**

$$\det \frac{\partial(u, v)}{\partial(x, y)}$$

as a function of (u, v) and hence find the function $\varphi(u, v)$ such that the **area element** dA in terms of the (u, v) coordinates is given by $dA = \varphi(u, v)\, dudv$.

QUESTION 8

The rectangular Cartesian coordinates (x, y, z) and the cylinderical polar coordinates (r, θ, z) for $\boldsymbol{R}^3$ are related by

$$x = r\,\cos\theta, \quad y = r\,\sin\theta \quad \text{and} \quad z = z.$$

Find the **volume element** dV in cylinderical polar coordinates for $\boldsymbol{R}^3$.

QUESTION 9

Sketch the region Ω in the (x, y)-plane which is determined by the inequalities:

$$1 - x \;\leq\; y \;\leq\; x - 1 \quad \text{and} \quad 1 \;\leq\; x \;\leq\; \sqrt{9 + y^2}.$$

Define the transformation (i.e. **change of variables**) $(x, y) \mapsto (u, v)$ by

$$u = x + y, \qquad v = x^2 - y^2$$

and sketch the image of Ω in the (u, v)-plane under this transformation.