MS321 Tutorial 1, Question 2

2. Consider the finite set D_4 of mappings from $\mathbb{R}^2$ to itself given by the 8 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Form the multiplication table for D_4 under the operation of composition of mappings or multiplication of the corresponding matrices.

Solution: Give the mappings letter names. The first is the identity. Let's call it I. As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $I(x, y) = (x, y)$. The second is reflection in the Y axis. Let's call it Y. As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $Y(x, y) = (x, -y)$. The third is reflection in the X axis. Lets call it X. As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $X(x, y) = (-x, y)$. (Skip the fourth for the moment.) The fifth is reflection in the line $y = x$. Lets call it P (for positive slope). As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $P(x, y) = (y, x)$. Similarly the eight is reflection in the line $y = -x$. Lets call it N (for negative slope). As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $N(x, y) = (-y, -x)$. The seventh is rotation through 90 degrees anticlockwise. Lets call it R (for rotation). As a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ its formula is $R(x, y) = (-y, x)$. Check that the fourth is R^2 and the sixth is R^3. Now the table should be (The entry in row a and column b is the product ab.)

	e	R	R^2	R^3	X	Y	P	N
e	e	R	R^2	R^3	X	Y	P	N
R	R	R^2	R^3	e	P	N	Y	X
R^2	R^2	R^3	e	R	Y	X	N	P
R^3	R^3	e	R	R^2	N	P	X	Y
X	X	N	Y	P	e	R^2	R^3	R
Y	Y	P	X	N	R^2	e	R	R^3
P	P	X	N	Y	R	R^3	e	R^2
N	N	Y	P	X	R^3	R	R^2	e

Here it might help to use the fact that a product of reflections is a rotation through twice the angle between the reflection lines so that

$$R = XN = PX = YP = NY$$

$$R^2 = XY = PN = YX = NP$$

$$R^3 = XP = PY = YN = NX$$

Cancellation gives

$$N = XR, X = PR, \text{etc.},$$

The same computation can be done with matrix multiplication. Use the fact that the columns of the matrices are very simple $\left(\pm\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \pm\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ and the fact that the ith column of AB is A times the ith column of B which in turn is the linear combination of the columns of A with coefficients given by the entries of the ith column of B. Thus, for example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}.$$