MS321 Tutorial 2, question 1

1. Show that any permutation in S_n can be expressed as a product of the elements from the set of transpositions of adjacent numbers

$$\{(1,2),(2,3),(3,4),\ldots,(n-1,n)\}.$$

(Example: For $n=4$, the transpositions are $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, while the transpositions of adjacent numbers are $\{(1,2),(2,3),(3,4)\}$)
(Hint: We already know that any permutation in S_n can be expressed as a product of general transpositions. Now write a general transposition in terms of transpositions of adjacent numbers.)

Since we know that any permutation in S_n can be expressed as a product of general transpositions we just need to show that a general transposition (i,j) can be expressed as a product transpositions of adjacent numbers.

We begin with the simplest case: $(1,3) \in S_3$. Can we swap 1 with 3 using a combination of swapping 1 with 2 and swapping 2 with 3? We begin by swapping 1 with 2. Now the first element is in the second place and swapping 2 with 3 will put it in the third place which is where we want it. Let us see what we have.

$$(2,3)\circ(1,2) = (1,3,2)$$

This is since

$$\begin{aligned}(2,3)\circ(1,2)(1) &= (2,3)(2) = 3,\\ (2,3)\circ(1,2)(2) &= (2,3)(1) = 1,\\ (2,3)\circ(1,2)(3) &= (2,3)(3) = 2.\end{aligned}$$

This is not quite right. 1 is where we want it. 2 and 3 are not. However they are in the first two positions so swapping 1 with 2 will put them in the correct places. Thus

$$(1,2)\circ(2,3)\circ(1,2) = (1,3)$$

It is worth following the elements through the compositions:

$$\begin{aligned}(1,2)\circ(2,3)\circ(1,2)(1) &= (1,2)\circ(2,3)(2) = (1,2)(3) = 3,\\ (1,2)\circ(2,3)\circ(1,2)(2) &= (1,2)\circ(2,3)(1) = (1,2)(1) = 2,\\ (1,2)\circ(2,3)\circ(1,2)(3) &= (1,2)\circ(2,3)(3) = (1,2)(2) = 1.\end{aligned}$$

1 went to 2, then to 3 and stayed there. 2 went to 1, stayed there and then went back to 2. 3 stayed where it was, then went to 2 and then to 1.

To produce $(1,4)$ you could start with $(3,4)\circ(2,3)\circ(1,2) = (1,4,3,2)$ which uts 1 is the correct place. Now 4 is in the third place and will be sent to the first place by $(1,2)\circ(2,3)$, so try that:

$$(1,2)\circ(2,3)\circ(3,4)\circ(2,3)\circ(1,2) = (1,4).$$

Again we can follow the elements through the compositions:

$$1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 4 \mapsto 4$$

$$2 \mapsto 1 \mapsto 1 \mapsto 1 \mapsto 1 \mapsto 2$$

$$3 \mapsto 3 \mapsto 2 \mapsto 2 \mapsto 3 \mapsto 3$$

$$4 \mapsto 4 \mapsto 4 \mapsto 3 \mapsto 2 \mapsto 1$$

The same kind of reasoning also gives $(2,3)\circ(1,2)\circ(2,3) = (1,3)$ and $(3,4)\circ(2,3)\circ(1,2)\circ(2,3)\circ(3,4) = (1,4)$. (Sort out the last element first, then the first.)

Now for the hard part: writing a general proof. The neatest way is probably to use induction.

For $i < k < j$ consider the product $(k,k+1)\circ(i,k)\circ(k,k+1)$:

$$\begin{aligned}
(k,k+1)\circ(i,k)\circ(k,k+1)(i) &= (k,k+1)\circ(i,k)(i) = (k,k+1)(k) = k+1 \\
(k,k+1)\circ(i,k)\circ(k,k+1)(k) &= (k,k+1)\circ(i,k)(k+1) = (k,k+1)(k+1) = k \\
(k,k+1)\circ(i,k)\circ(k,k+1)(k+1) &= (k,k+1)\circ(i,k)(k) = (k,k+1)(i) = i
\end{aligned}$$

so that $(k,k+1)\circ(i,k)\circ(k,k+1) = (i,k+1)$. Applied repeatedly, this gives

$$(i,j) = (j,j-1)\circ(j-1,j-2)\circ\ldots\circ(i+2,i+1)\circ(i,i+1)\circ(i+2,i+1)\circ\ldots\circ(j-1,j-2)\circ(j,j-1).$$