MS321 Algebra, tutorial 7

1. Determine all the homomorphisms fro $\mathbf{Z}_{15}$ to $\mathbf{Z}_{18}$.

Proceed as in the example in class. Let $f : \mathbb{Z}_{15} \to \mathbb{Z}_{18}$ be a homomorphism. We saw that f is determined by $f(1)$ since $\mathbb{Z}_{15}$ is cyclic with generator 1 and f is a homomorphism. However, in $\mathbb{Z}_{18}$,

$$0 = f(0) = f(15) = (15)f(1).$$

This means $(15)f(1) = (18)k$ in $\mathbb{Z}$. Dividing by the gcd of 15 and 18 gives $5f(1) = 6k$. Thus $5f(1)$ is a multiple of 6 and hence, $f(1)$ is a multiple of 6, by Q5 of tutorial 3. Hence $f(1) \in \{6, 12, 0\}$ giving 3 homomorphisms.

2. The group $\mathbb{Z}_{30}^*$ has 8 elements. There is a possibility it is isomorphic to one of $\mathbb{Z}_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Show that is not isomorphic to either.

$\mathbb{Z}_{30}^* = \{1, 7, 11, 13, 17, 19, 23, 29\}$. Consider the cyclic subgroup generated by 7.

$$\langle 7 \rangle = \{7, 19, 13, 1\}$$

($7^2 = 49 = 30 + 19$, $7^3 = 7(19) = 133 = 4(30) + 13$,
$7^4 = 7(13) = 91 = 3(30) + 1$) Since 7 has order 4, $\mathbb{Z}_{30}^*$ cannot be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which has no elements of order 4. To exclude the cyclic group we show that $\mathbb{Z}_{30}^*$ has no elements of order 8. We know that 1 has order 1 and $29 = 30 - 1$ has order 2. From the $\langle 7 \rangle$ computation we see that 7 has order 4, but also that 19 has order 2 and 13 has order 4. However, $23 = 30 - 7$ so that $23^4 = (30 - 7)^4 = 7^4$ up to multiples of 30. Thus $23^4 = 1$. Similarly, $17^4 = 1$ and $11^2 = 1$. That's them all! No elements of order 8.

3. Recall that for $\sigma = (i_1, i_2, \ldots, i_k)$ a k-cycle in S_n and $\tau \in S_n$,

$$\tau \circ \sigma \circ \tau^{-1} = (\tau(i_1), \tau(i_2), \ldots, \tau(i_k)).$$

Apply this to the case $\sigma = (i, i + 1)$ and $\tau = (i + 1, j)$ for $j > i + 1$ to get a new proof of the result in Q1 of Tutorial 2.

$\tau\sigma\tau^{-1} = (i, j)$

4. Suppose G is a group and define the set

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } \ g \in G\},$$

that is, the subset of G consisting of those elements which commute with all elements of G. Show that $Z(G)$ is a subgroup of G. Show that $Z(G)$ is normal in G.

(a) The element $e \in Z(G)$ since, for all $g \in G$, $ge = g$ and $eg = g$ so that $eg = ge$.
(b) Suppose $x, y \in Z(G)$ and let $g \in G$. Then

$$(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy),$$

where the second equality follows from $y \in Z(G)$ and the fourth equality follows from $x \in Z(G)$. Thus $xy \in Z(G)$.
(c) Suppose $x \in Z(G)$ and let $g \in G$. Then

$$x^{-1}g = (g^{-1}x)^{-1} = (xg^{-1})^{-1} = gx^{-1},$$

where the second equality follows from $x \in Z(G)$. Thus $x^{-1} \in Z(G)$.
Finally, if $x \in Z(G)$ and $g \in G$, then $g^{-1}xg = g^{-1}gx = x \in Z(G)$, so that $Z(G)$ is normal in G.