MS321 Algebra, tutorial 8

1. Suppose m and n are positive integers and define

$$\phi : \mathbb{Z} \to (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) : k \to (k + n\mathbb{Z}, k + m\mathbb{Z}).$$

Show that ϕ is a homomorphism and compute $\ker(\phi)$. Deduce that $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(mn)\mathbb{Z}$ when m and n are coprime.

$$\begin{aligned}
\phi(k + l) &= (k + l + n\mathbb{Z}, k + l + m\mathbb{Z}) \\
&= (k + n\mathbb{Z} + l + n\mathbb{Z}, k + m\mathbb{Z} + l + m\mathbb{Z}) \\
&= (k + n\mathbb{Z}, k + m\mathbb{Z}) + (l + n\mathbb{Z}, l + m\mathbb{Z}) \\
&= \phi(k) + \phi(l)
\end{aligned}$$

$$\begin{aligned}
k \in \ker(\phi) &\Leftrightarrow \phi(k) = (n\mathbb{Z}, m\mathbb{Z}) \\
&\Leftrightarrow (k + n\mathbb{Z}, k + m\mathbb{Z}) = (n\mathbb{Z}, m\mathbb{Z}) \\
&\Leftrightarrow k \in n\mathbb{Z} \text{ and } k \in m\mathbb{Z}
\end{aligned}$$

Thus $\ker(\phi) = l\mathbb{Z}$ where $l = \text{lcm}(m, n)$. In the case where m and n are coprime, $l = mn$ and $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(mn)\mathbb{Z}$ will follow from the first isomorphism theorem provided ϕ is onto. However, m and n coprime means there are integers a and b with $am + bn = 1$. Now for any $(p + n\mathbb{Z}, q + m\mathbb{Z}) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$ we have

$$\begin{aligned}
(p + n\mathbb{Z}, q + m\mathbb{Z}) &= (p(1) + n\mathbb{Z}, q(1) + m\mathbb{Z}) \\
&= (p(am + bn) + n\mathbb{Z}, q(am + bn) + m\mathbb{Z}) \\
&= (pam + n\mathbb{Z}, qbn + m\mathbb{Z}) \\
&= (pam + qbn + n\mathbb{Z}, qbn + pam + m\mathbb{Z}) \\
&= \phi(pam + qbn)
\end{aligned}$$

2. Suppose $H < G$ and $|G| = 2|H|$. Show that $H \lhd G$. That is, if a subgroup contains half the elements of a group the subgroup has to be normal. Hint: Use the $gH = Hg$ characterisation of normality.

If $|H| = (1/2)|G|$ then there are two left cosets of H in G and two right cosets of H in G. One of these cosets is H so the other is $G - H$ for both left and right cosets.

3. Suppose that H is a normal subgroup of G. Show that G/H is abelian if and only if

$$g_1 g_2 g_1^{-1} g_2^{-1} \in H, \text{ for any } \quad g_1, g_2 \in G.$$

Replacing g_1 by g_1^{-1} and g_2 by g_2^{-1} gives $g_1^{-1} g_2^{-1} g_1 g_2 \in H$ as the second condition. However

$$\begin{aligned} g_1^{-1} g_2^{-1} g_1 g_2 \in H &\Leftrightarrow (g_2 g_1)^{-1} g_1 g_2 \in H \\ &\Leftrightarrow (g_1 g_2)H = (g_2 g_1)H \\ &\Leftrightarrow g_1 H g_2 H = g_2 H g_1 H \end{aligned}$$

which is equivalent to G/H being abelian.

4. For each of the following examples compute the multiplication table for G/H:

(a) Let G be the group with multiplication table.

	e	a	b	c	x	p	q	r
e	e	a	b	c	x	p	q	r
a	a	x	c	q	p	e	r	b
b	b	r	x	a	q	c	e	p
c	c	b	p	x	r	q	a	e
x	x	p	q	r	e	a	b	c
p	p	e	r	b	a	x	c	q
q	q	c	e	p	b	r	x	a
r	r	q	a	e	c	b	p	x

The subgroup H generated by the element x is normal in G.

$H = \{e, x\}$, $aH = \{a, p\}$, $bH = \{b, q\}$, $cH = \{c, r\}$ are the cosets. The table is

	H	aH	bH	cH
H	H	aH	bH	cH
aH	aH	H	cH	bH
bH	bH	cH	H	aH
cH	cH	bH	aH	H

Here we have used

$$aa = x \in H, ab = c \in cH, ac = q \in bH,$$

$$ba = r \in cH, bb = x \in H, bc = a \in aH,$$

$$ca = b \in bH, cb = p \in aH, cc = x \in H.$$

(b) Let G be the group with multiplication table.

	e	x	x^2	x^3	y	yx	yx^2	yx^3
e	e	x	x^2	x^3	y	yx	yx^2	yx^3
x	x	x^2	x^3	e	yx	yx^2	yx^3	y
x^2	x^2	x^3	e	x	yx^2	yx^3	y	yx
x^3	x^3	e	x	x^2	yx^3	y	yx	yx^2
y	y	yx	yx^2	yx^3	e	x	x^2	x^3
yx	yx	yx^2	yx^3	y	x	x^2	x^3	e
yx^2	yx^2	yx^3	y	yx	x^2	x^3	e	x
yx^3	yx^3	y	yx	yx^2	x^3	e	x	x^2

The subgroup H generated by the element x^2 is normal in G.

$H = \{e, x^2\}$, $xH = \{x, x^3\}$, $yH = \{y, yx^2\}$, $yxH = \{yx, yx^3\}$ are the cosets. The table is

	H	xH	yH	yxH
H	H	xH	yH	yxH
xH	xH	H	yxH	yH
yH	yH	yxH	H	xH
yxH	yxH	yH	xH	H

Here we have used

$$xx = x^2 \in H, xy = yx \in yxH, xyx = y \in yH,$$

$$yx = yx \in yxH, yy = e \in H, yyx = x \in xH,$$

$$yxx = yx^2 \in yH, yxy = x \in xH, yxyx = x^2 \in H.$$

(c) $G = D_4$, $H = \langle R^2 \rangle$. The multiplication table for D_4 is shown below. (The entry in row a and column b is the product ab.)

	e	R	R^2	R^3	X	Y	P	N
e	e	R	R^2	R^3	X	Y	P	N
R	R	R^2	R^3	e	P	N	Y	X
R^2	R^2	R^3	e	R	Y	X	N	P
R^3	R^3	e	R	R^2	N	P	X	Y
X	X	N	Y	P	e	R^2	R^3	R
Y	Y	P	X	N	R^2	e	R	R^3
P	P	X	N	Y	R	R^3	e	R^2
N	N	Y	P	X	R^3	R	R^2	e

$H = \{e, R^2\}$, $RH = \{R, R^3\}$, $XH = \{X, Y\}$, $PH = \{P, N\}$ are the cosets. The table is

	H	RH	XH	PH
H	H	RH	XH	PH
RH	RH	H	PH	XH
XH	XH	PH	H	RH
PH	PH	XH	RH	H

Here we have used

$$RR = R^2 \in H, RX = P \in PH, RP = Y \in XH,$$

$$XR = N \in PH, XX = e \in H, XP = R^4 \in RH,$$

$$PR = X \in XH, PX = R \in RH, PP = e \in H.$$