MS321 Algebra, Tutorial 9

1. Up to isomorphism, how many abelian groups are there of order 42, 36, 37?

$42 = 2(3)7$ so only one group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{42}$.
$36 = 2^2 3^2$ so there will be four groups, depending on whether or not the 2's are split and whether or not the 3's are split. This gives

$$\mathbb{Z}_4 \times \mathbb{Z}_9,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9,\ \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

37 is prime so only one group $\mathbb{Z}_{37}$.

2. If G is a finite abelian group and p is a prime factor of $|G|$, prove that G has an element of order p.

We know $G \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \ldots \times \mathbb{Z}_{p_n^{k_n}}$ so that $|G| = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$. Since p is a divisor of $|G|$, one of these p_i must be p. Thus $\mathbb{Z}_{p^k}$ is one of the factors with $k \geq 0$. If x is a generator of this factor then $p^{k-1}x$ is an element of order p.

3. Use the structure theorem for finite abelian groups to prove that every abelian group of order 72 has at least one element of order 6.

Since $72 = 2^3 3^2$, the possible structures are

$$\mathbb{Z}_8 \times \mathbb{Z}_9,$$

$$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9,$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9,$$

$$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3,$$

$$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3,$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3,$$

The following are elements of order 6 in the respective groups :

$$(4,3),(1,0,3),(1,0,0,3),(4,1,0),(1,0,1,0),(1,0,0,1,0),$$

4. What is the structure of the abelian groups $\mathbb{Z}_{36}^*$ and $\mathbb{Z}_{21}^*$?

$36 = 2^2 3^2$ so that $\mathbb{Z}_{36}^*$ has $36 - 18 - 12 + 6 = 12$ elements. Thus the group is isomorphic to one of

$$\mathbb{Z}_4 \times \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

$$\mathbb{Z}_{36}^* = \{1,5,7,11,13,17,19,23,25,29,31,35\}$$

We consider orders of elements.

$$\langle 5 \rangle = \{5,25,17,13,29,1\}$$

We deduce that 17 has order 2. However, 35 also has order 2. This means that $\mathbb{Z}_{36}^*$ is not isomorphic to the cyclic group $\mathbb{Z}_4 \times \mathbb{Z}_3$ which only has one element of order 2. Thus $\mathbb{Z}_{36}^*$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$21 = 3(7)$ is a product of two distinct primes, so that $\mathbb{Z}_{21}^*$ has $(3-1)(7-1) = 12$ elements. Thus the group is isomorphic to one of

$$\mathbb{Z}_4 \times \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

$$\mathbb{Z}_{21}^* = \{1,2,4,5,8,10,11,13,16,17,19,20\}$$

We consider orders of elements.

$$\langle 2 \rangle = \{2,4,8,16,11,1\}$$

We deduce that 8 has order 2. However, 20 also has order 2. This means that $\mathbb{Z}_{21}^*$ is not isomorphic to the cyclic group $\mathbb{Z}_4 \times \mathbb{Z}_3$ which only has one element of order 2. Thus $\mathbb{Z}_{21}^*$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.