MS321 Algebra, tutorial 8

1. Suppose m and n are positive integers and define

$$\phi : \mathbb{Z} \to (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) : k \to (k + n\mathbb{Z}, k + m\mathbb{Z}).$$

Show that ϕ is a homomorphism and compute $\ker(\phi)$. Deduce that $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(mn)\mathbb{Z}$ when m and n are coprime.

2. Suppose $H < G$ and $|G| = 2|H|$. Show that $H \triangleleft G$. That is, if a subgroup contains half the elements of a group the subgroup has to be normal. Hint: Use the $gH = Hg$ characterisation of normality.

3. Suppose that H is a normal subgroup of G. Show that G/H is abelian if and only if

$$g_1 g_2 g_1^{-1} g_2^{-1} \in H, \text{ for any } \quad g_1, g_2 \in G.$$

4. For each of the following examples compute the multiplication table for G/H:

(a) Let G be the group with multiplication table.

	e	a	b	c	x	p	q	r
e	e	a	b	c	x	p	q	r
a	a	x	c	q	p	e	r	b
b	b	r	x	a	q	c	e	p
c	c	b	p	x	r	q	a	e
x	x	p	q	r	e	a	b	c
p	p	e	r	b	a	x	c	q
q	q	c	e	p	b	r	x	a
r	r	q	a	e	c	b	p	x

The subgroup H generated by the element x is normal in G.

(b) Let G be the group with multiplication table.

	e	x	x^2	x^3	y	yx	yx^2	yx^3
e	e	x	x^2	x^3	y	yx	yx^2	yx^3
x	x	x^2	x^3	e	yx	yx^2	yx^3	y
x^2	x^2	x^3	e	x	yx^2	yx^3	y	yx
x^3	x^3	e	x	x^2	yx^3	y	yx	yx^2
y	y	yx	yx^2	yx^3	e	x	x^2	x^3
yx	yx	yx^2	yx^3	y	x	x^2	x^3	e
yx^2	yx^2	yx^3	y	yx	x^2	x^3	e	x
yx^3	yx^3	y	yx	yx^2	x^3	e	x	x^2

The subgroup H generated by the element x^2 is normal in G.

(c) $G = D_4$, $H = \langle R^2 \rangle$. The multiplication table for D_4 is shown below. (The entry in row a and column b is the product ab.)

	e	R	R^2	R^3	X	Y	P	N
e	e	R	R^2	R^3	X	Y	P	N
R	R	R^2	R^3	e	P	N	Y	X
R^2	R^2	R^3	e	R	Y	X	N	P
R^3	R^3	e	R	R^2	N	P	X	Y
X	X	N	Y	P	e	R^2	R^3	R
Y	Y	P	X	N	R^2	e	R	R^3
P	P	X	N	Y	R	R^3	e	R^2
N	N	Y	P	X	R^3	R	R^2	e