An anisotropic model for global climate data

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Gaussian process geostatistics

Given $y_1, \cdots, y_n \in \mathbb{R}$ measured at locations $x_1, \cdots, x_n \in S$, which we want to interpolate to the whole space S, the classical approach in geostatistics could be sketched as follow:

- Fit a Gaussian process model $Y_{x \in S}$ to the data.
- Interpolate the data at a new location x^* with the Kriging estimator

$$\hat{Y}(x^*) = \mathbb{E}(Y(x^*) | Y(x_1) = y_1, \cdots, Y(x_n) = y_n). \tag{1}$$

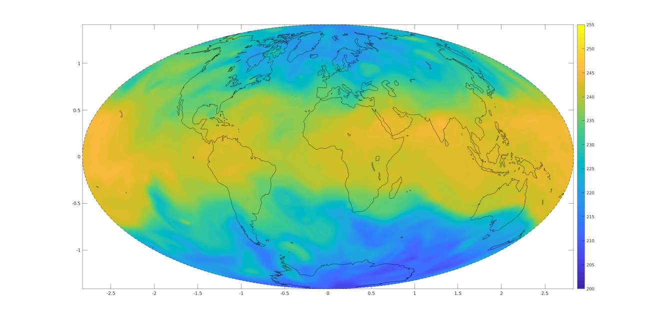
- Enjoy the benefits of a statistical model.

In order to do this, we need rich classes of Gaussian processes indexed by our space S.

Global data: From the Euclidean plane to the sphere

- To handle global climate data, we need to consider (at least) Gaussian processes indexed by the sphere.
- The case of isotropic processes on the sphere, (which covariance is a function of the distance) is now well understood, and in particular we have equivalents of the classical Euclidean Gaussian Processes, such as Matèrn and exponential kernels.
  (See Gneiting 2013 [1], Porcu et al. 2018 [7], Jeong et al. 2017 [3] and references within for the state of the art.)

However global climate data is not isotropic

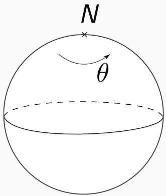


- Climate data exhibits complex anisotropic behaviors.
- In particular it is clear that climate data is correlated at longer ranges in the longitudinal direction.

Anisotropic models in the literature

Jones 1963 [4] proposes to address this issue and defines axially symmetric Gaussian processes, which are stationary only in longitude variable, that is to say their covariance verifies

$$K_X(x, y) = F(\theta_x - \theta_y \mod 2\pi, \varphi_x, \varphi_y). \quad (2)$$



Furthermore an axially symmetric Gaussian process is said to be *latitudinally reversible* if

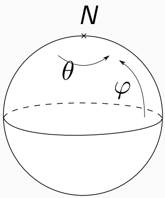
$$F(\theta_x - \theta_y \mod 2\pi, \varphi_x, \varphi_y) = F(\theta_x - \theta_y \mod 2\pi, \varphi_y, \varphi_x). \quad (3)$$

Anisotropic models in the literature 2

- Jones 1963 [4] gives a general decomposition of axially symmetric covariances on the spherical harmonic basis.
- Stein 2007 [9] truncates it to a finite order to carry on a statistical analysis of a Total Ozone dataset at a global scale.
- Jun and Stein 2007 [5, 6] differentiate isotropic processes to obtain axially symmetric models.
- Huang et al. 2012 [2] consider products of separated covariances on latitudes and longitudes.
- Recently Porcu et al. 2019 [8] proposed to modify variograms of isotropic covariances to obtain axially symmetric analogues.

Conventions on the sphere

- We consider the sphere $\mathbb{S}^1$ of radius 1.
- To a given point x on the sphere are associated its longitude $\theta_x \in [-\pi, \pi]$ and latitude $\varphi_x \in [-\pi/2, \pi/2]$, in radians.
- We will use the geodesic distance on the sphere which is given by the formula



$$d(x, y) = \cos^{-1}\left(\sin\varphi_1 \cdot \sin\varphi_2 + \cos\varphi_1 \cdot \cos\varphi_2 \cdot \cos\left(\theta_2 - \theta_1\right)\right). \quad (4)$$

- Notice that the longitude is not well-defined at the poles but that (4) is consistent for any chose values.

A naive approach... and a problem at the poles

Starting from a valid isotropic kernel, like the *exponential kernel*

$$K(x, y) = \sigma^2 e^{-\frac{d(x,y)}{r}}, \tag{5}$$

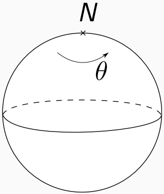
a simple idea to obtain anisotropy is to separate the latitude and longitude variables:

$$K_{sep}(\theta_x, \theta_y, \varphi_x, \varphi_y) = \sigma^2 e^{-\frac{|\theta_x - \theta_y|}{r_\theta}} e^{-\frac{|\varphi_x - \varphi_y|}{r_\varphi}}. \tag{6}$$

- K_{sep} is valid as a product of valid kernels.
- But since longitude is not well-defined at the poles we cannot write

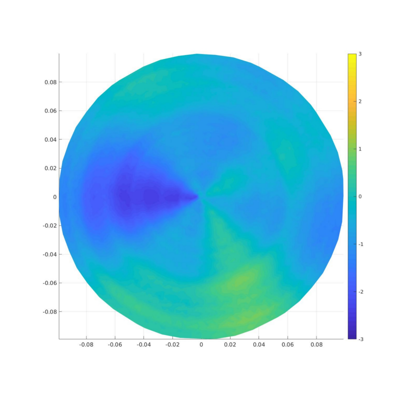
  $$K_{sep}(\theta_x, \theta_y, \varphi_x, \varphi_y) = K(x, y):$$

  the kernel is not defined at the poles!



A closer look at the poles

- From the point of view of applications, having a random field that is not defined in two points is not an issue.



- But... the random fields indexed by $\mathbb{S}^2 \setminus \{N, S\}$ that we obtain exhibit singular behaviour in the neighborhood of the poles.

Our approach

Starting from the same valid isotropic kernel, (like the *exponential kernel*)

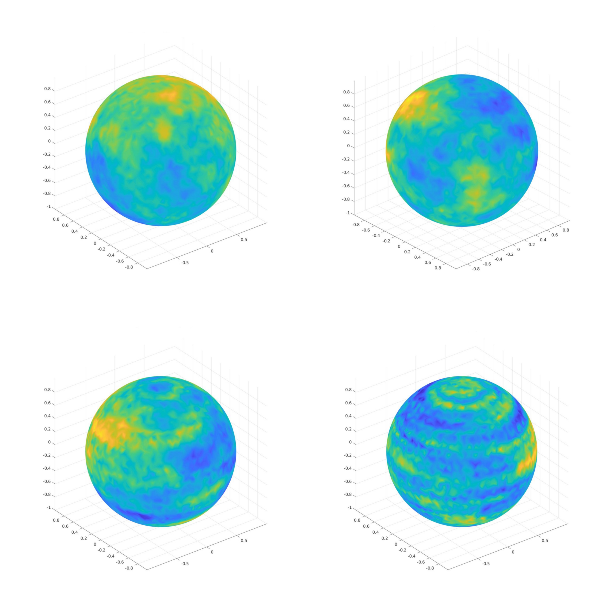
$$K(x,y) = \sigma^2 e^{-\frac{d(x,y)}{r}}, \tag{7}$$

our approach is to add decorrelation in the latitudinal direction by multiplying by another kernel

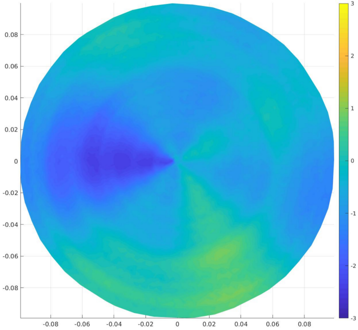
$$K_{new}(x,y) = \sigma^2 e^{-\frac{d(x,y)}{r_{iso}}} \cdot \; e^{-\left(\frac{|\varphi_x - \varphi_y|}{r_\varphi}\right)^2}. \tag{8}$$

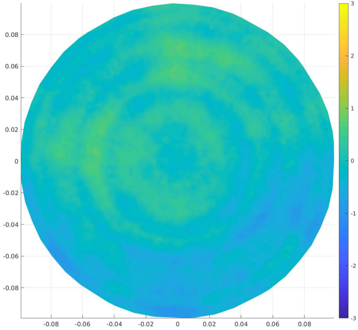
- Like previously, K_{new} is valid as a product of valid kernels.
- This time K_{new} is defined over $\mathbb{S}^2 \times \mathbb{S}^2$.
- K_{new} is *axially symmetric* (and longitudinally reversible) by construction.

Simulation of K_{new} with fixed r_{iso} and increasing r_{φ} parameters



A closer look at the North pole, K_{sep} VS K_{new}





As expected our proposed Gaussian Process is continuous at the poles.

A new class of axially symmetric covariances

Theorem (Continuous axially symmetric covariances)

Let K_{iso} be an isotropic covariance on the sphere and K_φ be a covariance on $[-\pi, \pi]$.

The kernel defined by

$$K(x, y) = K_{iso}(x, y) \cdot K_\varphi(\varphi_x, \varphi_y) \tag{9}$$

is a latitudinally reversible, axially symmetric covariance on the sphere.

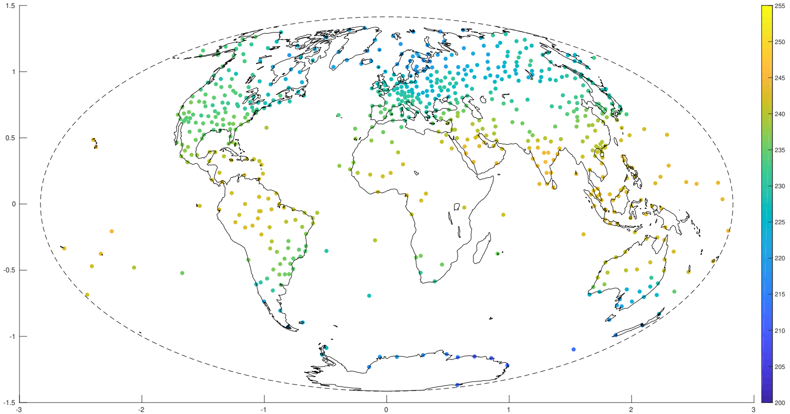
Furthermore, if K_{iso} and K_φ are continuous, K is continuous on the whole sphere, and as such, a Gaussian field with covariance K is continuous in L^2 sense, and has almost surely continuous trajectories.

Choice of the training dataset

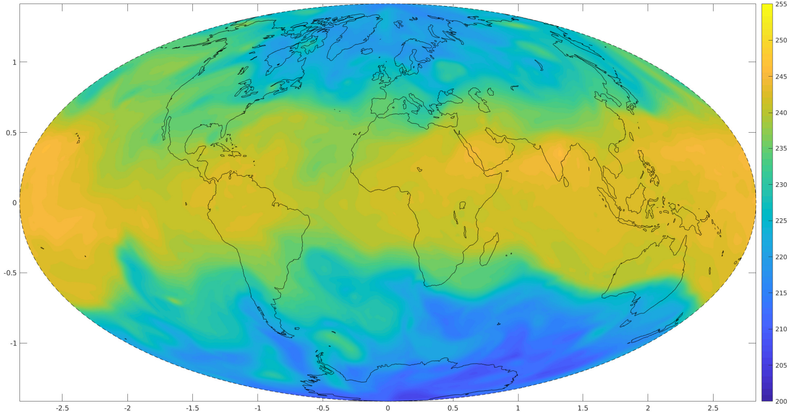
- We use ERA-Interim data, which is a reanalysis of global atmospheric data from 1979 produced by the ECMWF[1].
- We arbitrarily focus on temperature on January 27th 1999 at noon, at the altitude corresponding to the midrange pressure level of $300hPa$. We sample the data at the locations of radiosonde stations.
- ERA-Interim data is chosen because its completeness and physical coherence allow for virtually any further development, RAOB locations for the likelihood of the application.

[1] European Centre for Medium-Range Weather Forecasts

The training dataset: ERA-Interim at RAOB locations



The whole temperature field for the same day and altitude

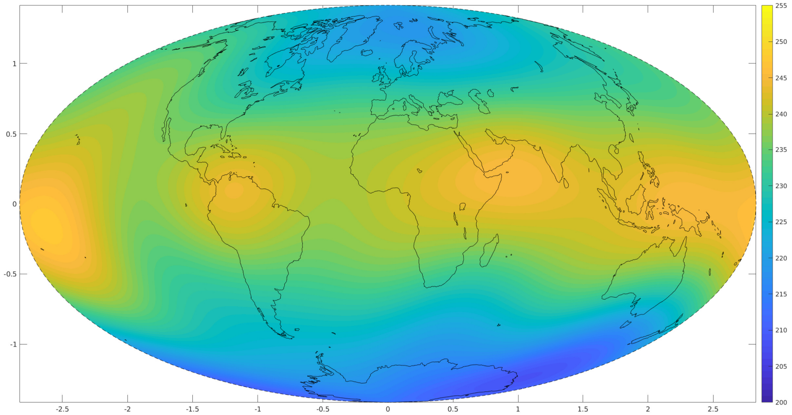


Detrending

We do not pretend to tackle the whole anisotropic behavior of the dataset with our model.

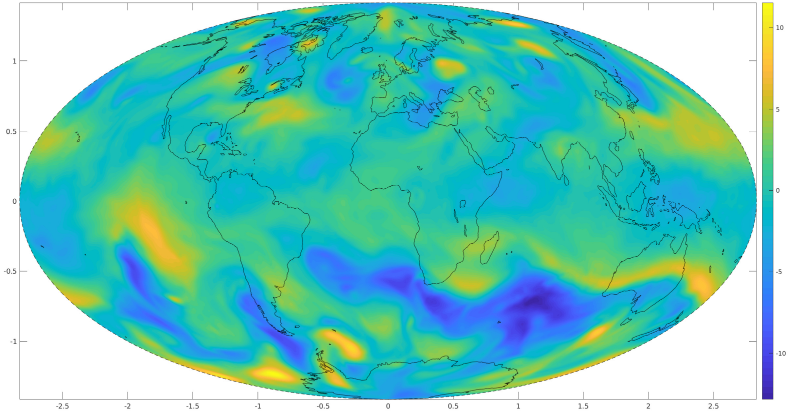
- Following Stein 2007 [9], we detrend with a truncated sum of spherical harmonics, fitted using a least square method.
- The order of truncation is optimized by minimizing the risk on the whole temperature field. A total of 36 basis functions is used.

Spherical trend



The obtained trend is smooth and accounts well for the large scale behaviour of the data.

ERA-Interim residuals after spherical detrending



The residuals seem to still exhibits directional anisotropic at a medium scale.

Fitting the model

- The Gaussian process and a Gaussian white noise (nugget) term are fitted to the residuals by maximizing the log-likelihood.
- We compare the performances of the three covariance kernels:

$$K_{iso}(x, y) = \sigma^2 e^{-\frac{d(x,y)}{r_1}}, \tag{10}$$

$$K_{sep}(\theta_x, \theta_y, \varphi_x, \varphi_y) = \sigma^2 e^{-\frac{|\theta_x - \theta_y|}{r_1}} e^{-\frac{|\varphi_x - \varphi_y|}{r_2}}. \tag{11}$$

$$K_{new}(x, y) = \sigma^2 e^{-\frac{d(x,y)}{r_1}} \cdot \; e^{-\left(\frac{|\varphi_x - \varphi_y|}{r_2}\right)^2}. \tag{12}$$

using the *Aikake Information Criterion* (number of parameters minus log-likelihood).

Numerical results

	K_{iso}	K_{new}	K_{sep}
initial σ	1	1	1
initial range r_1 (km)	3000	3000	3000
initial range r_2 (km)		1000	3000
σ	2.29	2.21	1.99
range r_1 (km)	1284	1780	796
range r_2 (km)		803	1933
nugget standard deviation	0.02	0.02	0.02
Aikake Information Criterion	1202	1159	1191

- Our covariance outperforms both isotropic and "naive anisotropic" alternatives.
- The estimated ranges and parameters are consistent.

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